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Collapsing a Series

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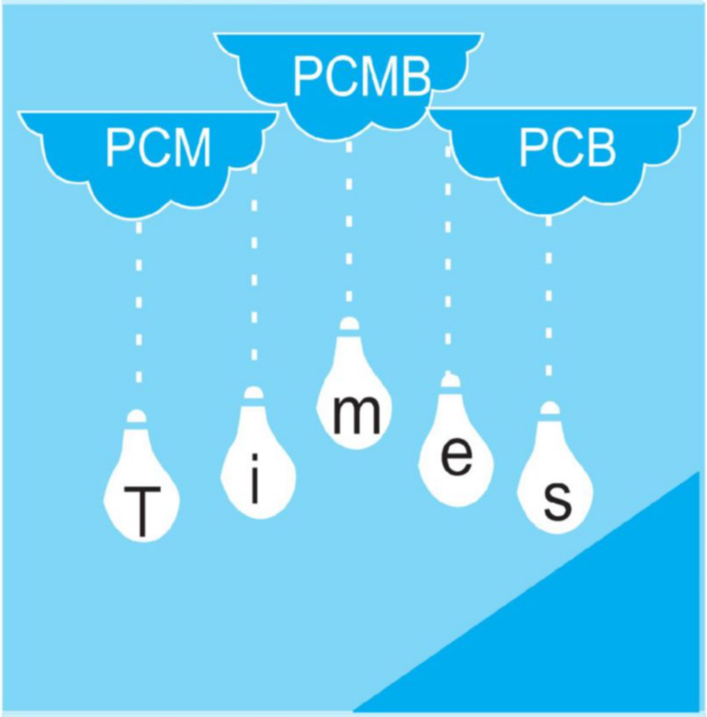

CONTENTS

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Collapsing a Series

Concept of the month

This column is aimed at preparing students for all competitive exams like JEE, BITSAT etc. Every concept has been designed by highly qualified faculty to cater to the needs of the students by discussing the most complicated and confusing concepts in Mathematics.

By. **DHANANJAYA REDDY THANAKANTI**
(Bangalore)

Abstract: This article highlights a technique for evaluating sums or products that we call Collapsing Series in which we write terms in way that they cancel each other. In this article , this technique is illustrated by a series of problems starting off with some simple ones in arithmetic, then some in trigonometry, inequalities and finally ending with a class of problems.

Collapsing is a techinique for finding the sum or product of certain series.The other name of this technique is *Telescoping*, which is most well-known name. Collapsing series (Telescoping series) are series whose terms will stack together and cancel themselves out. Just like the telescopes that some pirates used, the telescopes that these sums were named after those telescopes you could stack together and stuff into your pocket very easily. In the same way, it's way easier to find the value of what's left of a telescoping sum after you collapse it than the entire telescope. In this regard, telescoping sums can be viewed as an example of a huge class of math problems using the top-down approach; breaking a problem into simpler components often reveals important patterns.

Sum

Evaluating a telescoping sum often requires the knowledge of how to decompose fractions into partial fractions. The technique of partial fraction decomposition also arises in the integration of rational functions.

The best way to explain how to telescope a series and what series are telescopic is to show some examples.

Example.

$$1+2-2+3-3+4-4+...+10-10+11$$

This becomes

$$1+(2-2)+(3-3)+(4-4)+...+(10-10)+11=1+11=12$$

Example. Evaluate

$$(1-2)+(2-3)+...(2013-2014)$$

Clearly one would realize that this is equivalent to $(-1)\times 2013=-2013.$ However, if we think about this using telescoping sums, we realize that the -2 and 2 cancel out, then -3 and 3, and so on until the -2013 and 2013. The ones that we are left with are 1 and $-2014,$ so the sum is $1-2014=-2013.$

Let's go to a bit more *challenging* one.

Example. Evaluate

$$(1+2-3-4)+(2+3-4-5)+...$$
$$+(97+98-99-100)$$

If one wants to compute out each parenthesis (each is equal to $k+(k+1)-(k+2)-(k+3)=-4$ and note that there are 97 parentheses, the sum would be $-4\times 97=-388.$

However, we could think about this with telescoping sums!

It is easy to notice that there are 97 parentheses. Consider the $1^{st}, 3^{rd}, \ldots, 97^{th}$ parenthesis. The 3 and the 4 in the first parenthesis cancels out with the 3 and the 4 in the third parenthesis. The 5 and the 6 in the third parenthesis cancels out with the 5 and 6 in the fifth parenthesis. We could continue doing this, until we are left with $1+2-99-100$.

With regards to the *even-ordered* parentheses, we are left with $2+3-98-99$. Adding up, we get

$$(1+2-99-100)+(2+3-98-99)$$
$$=-196+(-192)$$
$$=-388.$$

Sum of Reciprocals (or Something Close to it)

Example. Evaluate $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{9\cdot 10}$

Since we are trying to make this a telescoping sum, we try to express each as the difference of two fractions.

We know that $\frac{1}{k}-\frac{1}{k+1}=\frac{1}{k(k+1)}$. so the sum is equal to

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{9}-\frac{1}{10}\right)=\frac{1}{1}-\frac{1}{10}=\frac{9}{10}.$$

With this, we could generalize this to

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{k}-\frac{1}{k+1}\right)=\frac{k(k+1)-1}{k(k+1)}$$

Of course, the difference between the terms is not necessarily equal to 1.
For example, the following is true:

$$\frac{1}{k}-\frac{1}{k+n}=\frac{n}{k(k+n)}$$

If the common differences are all equal, then we could still represent the sum as a telescoping sum, and then the terms will cancel out.

Example. Evaluate $\frac{1}{(3)(5)}+\frac{1}{(5)(7)}+\ldots+\frac{1}{(97)(99)}$

We could write the sum as

$$\frac{\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\ldots+\left(\frac{1}{97}-\frac{1}{99}\right)}{2}=\frac{\left(\frac{1}{3}-\frac{1}{99}\right)}{2}=\frac{16}{99}.$$

Generalisation

If we have a sequence $a_0, a_1, a_2, \ldots,$ and we can find a function $f(n)$ such that

$a_n = f(n+1)-f(n)$ for every n, then

$$\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} f(i+1)-f(i) = f(n+1)-f(0)$$

(because all the other terms cancel).
But finding such a function explicitly may be difficult.

Clearly this method could be extended to products, with the numerators and denominators cancelling out.

Product

A collapsing series of product is a series where each term can be represented in a certain form, such that the multiplication of all of the terms results in massive cancellation of numerators and denominators.

The simplest form of a telescoping product occurs when $a_k = \frac{t_k}{t_k+1}$, in which case the product of these n terms is equal to

$$a_1 \times a_2 \times \ldots \times a_n = \frac{t_1}{t_2}\times\frac{t_2}{t_3}\times\ldots\times\frac{t_n}{t_{n+1}}=\frac{t_1}{t_{n+1}}.$$

Observe that the cancellation of each denominator with the numerator of the subsequent term allows us to arrive at the final result immediately.

Explanation of the Product Notation $\prod$

Before we get started with the calculation, it would be better to simplify the expression into a simple form so that it is easier for the readers to digest the information, than to write it in the long form $a_1 \times a_2 \times ... \times a_n$. This is because it will become an inconvenience to convey the same message especially when a shorter description is always preferred. We will be introducing a symbol $\prod$ to describe the product of such terms.

Let us begin with an introductory example of how telescoping product can be motivated. Consider the product

$$\prod_{k=1}^{9} \frac{k}{k+1} = \frac{1}{2} \times \frac{2}{3} \times ... \times \frac{9}{10}.$$

Let's try to find the product of first few terms one at a time:

$$\frac{1}{2} = \frac{1}{2}$$

$$\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$$

$$\frac{1}{3} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$$

$$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{1}{5}.$$

All the 4 equations represent the product of the first n fractions in the product $\prod_{k=1}^{9} \frac{k}{k+1}$. We notice that all the final values of these equations are in the form of $\frac{1}{n+1}$, where $n = 1, 2, 3, 4$.

Recall that we are essentially multiplying fractions, so if the numerator of one of the fraction is equal to the denominator of another fraction (both of which are non-zero), we can cancel them off in pairs and thus have

$$\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{3} = \frac{1}{3}.$$

Similarly, cancelling off the 2's and 3's in pairs shows that $\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{4} = \frac{1}{4}$. Analogously, we can compute the product in question by cancelling off almost all the numbers in pairs:

$$\prod_{k=1}^{9} \frac{k}{k+1} = \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{4} \times \frac{4}{\cancel{5}} \times \frac{\cancel{5}}{\cancel{6}} \times \frac{\cancel{6}}{\cancel{7}} \times \frac{\cancel{7}}{\cancel{8}} \times \frac{\cancel{8}}{\cancel{9}} \times \frac{\cancel{9}}{10} = \frac{1}{10}.$$

Explanation of how $\frac{T(n \pm k)}{T(n)}$ Works

How do we systematically identify which terms to cancel out with which? For example, by writing out the first few terms of the product below, we still are not able to cancel out any terms, right?

$$\prod_{m=1}^{9} \frac{m}{m+5} = \frac{1}{6} \times \frac{2}{7} \times \frac{3}{8} \times ... \times ... \times \frac{10}{5}$$

Well, the technique is actually the same, but we need a slight adjustment to show that there exist such terms that are being cancelled off, as shown below:

$$\prod_{m=1}^{9} \frac{m}{m+5} = \frac{1}{6} \times \frac{2}{\cancel{7}} \times \frac{3}{\cancel{8}} \times \frac{4}{\cancel{9}} \times \frac{5}{\cancel{10}} \times \frac{\cancel{6}}{11} \times \frac{\cancel{7}}{12} \times \frac{\cancel{8}}{13} \times \frac{\cancel{9}}{14} \times \frac{\cancel{10}}{5}$$

$$= \frac{1 \times 2 \times 3 \times 4 \times 5}{11 \times 12 \times 13 \times 14 \times 15} = \frac{1}{3003}.$$

Notice that for each of the fractions in the product $\prod_{m=1}^{10} \frac{m}{m+5}$, the numerator is always 5 less than the denominator, so only after the 5^{th} term are we able to see that the terms start to cancel out. So we can rewrite this as $\prod_{m=1}^{10} \frac{T(m-5)}{T(m)}$, where $T(m) = m + 5$.

With the fraction (in the product) written as a function of m, it will be easier to determine how a product telescopes.

To generalize all such telescoping products (products that are able to telescope, i.e. able to cancel out 1 or more terms in pairs), we define $\prod_{m} S(m)$ to be able to telescope if it can be

expressed in the form of $\prod \frac{T(m\pm k)}{T(m)}$ for some integer k. To illustrate this point, let us define two terms: backward cancellation and forward cancellation.

Backward Cancellation: Backward cancellation is the cancellation of terms in pairs in the case that the term(s) in the denominator of a fraction is(are) cancelled out with the term(s) in the numerator of a later fraction, where the product can be expressed as $\prod \frac{T(m+k)}{T(m)}$ for some positive integer k.

Forward Cancellation: Forward cancellation is the cancellation of terms in pairs in the case that the term(s) in the denominator of a fraction is(are) cancelled out with the term(s) in the numerator of a prior fraction, where the product can be expressed as $\prod \frac{T(m+k)}{T(m)}$ for some positive integer k.

Trigonometric Applications

Trigonometric identities (usually the angle addition identity) would make telescoping sums easy. The double-angle identity would also help.

Example. 1 Find $\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)$

Sol: First notice that the problem involves powers of 2, which seems especially interesting since $7=2^3-1$. That suggests that a simplification technique such as double angle may succeed. In order to apply the sine double angle relation $\sin(2\theta)=2\sin(\theta)\cos(\theta)$, we need both sines and cosines. The problem seems to conveniently leave out any sines, but let's try to introduce them on both sides. Let the desired product be P, and consider $P\left(\sin\left(\frac{\pi}{7}\right)\right)$. We have that

$$P\left(\sin\left(\frac{\pi}{7}\right)\right)=\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)$$

$$=\frac{1}{2}\sin\left(\frac{2\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)$$

$$=\frac{1}{2^2}\sin\left(\frac{4\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)$$

$$=\frac{1}{2^3}\sin\left(\frac{8\pi}{7}\right)$$

$$=-\frac{1}{8}\sin\left(\frac{\pi}{7}\right)$$

Example.2 Prove that $\frac{1}{\cos 0^\circ \cos 1^\circ}+\frac{1}{\cos 1^\circ \cos 2^\circ}+\ldots+\frac{1}{\cos 88^\circ \cos 89^\circ}=\frac{\cos 1^\circ}{\sin^2 1^\circ}$

Sol: We move one of the occurrences of the $\sin 1^\circ$ to the numerators, to get

$$\frac{\sin 1^\circ}{\cos 0^\circ \cos 1^\circ}+\frac{\sin 1^\circ}{\cos 1^\circ \cos 2^\circ}+\ldots+\frac{\sin 1^\circ}{\cos 88^\circ \cos 89^\circ}=\frac{\cos 1^\circ}{\sin 1^\circ}$$

Now we would want to see how is $\sin 1^\circ$ relevant to $\cos k^\circ$ and $\cos(k+1)^\circ$. We could use the difference identity, which is

$\sin(a-b)^\circ=\sin a^\circ \cos b^\circ-\cos a^\circ \sin b^\circ$.

Hence $\sin 1^\circ=\sin(k+1)^\circ \cos k^\circ-\cos(k+1)^\circ \sin k^\circ$.

Thus we can write each term as

$$\frac{\sin(k+1)^\circ \cos k^\circ-\cos(k+1)^\circ \sin k^\circ}{\cos k^\circ \cos(k+1)^\circ}$$

$$=\frac{\sin(k+1)^\circ}{\cos(k+1)^\circ}-\frac{\sin k^\circ}{\cos k^\circ}.$$

Summing up all of these from $k=0^\circ$ to $k=88^\circ$ gives that the sum is

$$\frac{\sin(89)^\circ}{\cos(89)^\circ}-\frac{\sin 0^\circ}{\cos^\circ}=\frac{\cos 1^\circ}{\sin 1^\circ}$$

The above ideas may be applied to inequalities and calculus.

Exercise

1. Find the value of $\frac{1}{3^2+1}+\frac{1}{4^2+2}+\frac{1}{5^2+3}+\ldots$

2. Determine the value of the sum

$$\frac{3}{1^2\cdot 2^2}+\frac{5}{2^2\cdot 3^2}+\frac{7}{3^2\cdot 4^2}+\ldots+\frac{29}{14^2\cdot 15^2}$$

3. Evalute the sum $\sum_{k=1}^{n}\frac{1}{(k+1)\sqrt{k}+k\sqrt{k+1}}$

4. The product $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{9^2}\right)\left(1-\frac{1}{10^2}\right)$

(a) $\frac{5}{12}$ (b) $\frac{1}{2}$ (c) $\frac{11}{20}$ (d) $\frac{2}{3}$

5. Find $\frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{4^2}{4^2-1}\cdots\frac{2006^2}{2006^2-1}$

6. Evaluate the infinite product: $\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdots$.

7. Evalute $\cos\theta\times\cos 2\theta\times\cos 4\theta\times\cos 8\theta\times\cos 16\theta$.

8. Evalute the sum $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{99\cdot 100}$.

9. Define $a_k=(k^2+1)k!$ and $b_k=a_1+a_2+a_3+\ldots+a_k$.

Let $\frac{a_{100}}{b_{100}}=\frac{m}{n}$ where m and n are relatively prime natural numbers. Find $n-m$.

10. $\sum_{r=1}^{44}\frac{1}{\cos r^0\cos(r+1)^0}=$

(a) $2\sqrt{2}\frac{\sin 44^0}{\sin 2^0}$ (b) $2\frac{\sin 44^0}{\sin 2^0}$

(c) $\sqrt{2}\frac{\sin 44^0}{\sin 2^0}$ (d) None of these

11. Evaluate $\sum_{r=1}^{n}\tan 2^{r-1}\theta\sec 2^r\theta$

12. Evaluate

$$\tan x\tan 2x+\tan 2x\tan 3x+\ldots.+\tan nx\tan(n+1)x.$$

13. $\cot^2 1^0-\sum_{r=1}^{88}\tan r^0\tan(r+1)^0$

(a) 89 (b) 88 (c) 90 (d) 91

14. Find the Sum of n terms

$$\sin^3\frac{\theta}{3}+3\sin^3\frac{\theta}{3^2}+3^2\sin^3\frac{\theta}{3^3}+\ldots$$

15. $S_n=\frac{1}{2\cos\theta}+\frac{1}{2^2\cos\theta\cos 2\theta}$

$$+\frac{1}{2^3\cos\theta\cos 2\theta\cos 2^2\theta}+\ldots.+\frac{1}{2^n\cos\theta\ldots..\cos 2^n\theta}$$

HINTS & SOLUTIONS

1.Sol: Each term takes the form

$$\frac{1}{n^2+(n-2)}=\frac{1}{(n+2)(n-1)}$$

Using the method of partial fractions, we can write (for some constant A, B)

$$\frac{1}{(n+2)(n-1)}=\frac{A}{n+2}+\frac{B}{n-1}$$

$$\Rightarrow \qquad A\cdot(n-1)+B\cdot(n-2)$$

Setting $n=1$ we get $B=\frac{1}{3}$, and similarly with $n=-2$ we get $A=-\frac{1}{3}$. Hence the sum becomes

$$\frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{3}-\frac{1}{6}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{5}-\frac{1}{8}\right)+\ldots$$

Thus, it telescopes, and the only terms that do not cancel produce a sum of $\frac{1}{3}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=\frac{13}{26}$.

2.Sol: The sum telescopes as

$$\left(\frac{1}{1^2}-\frac{1}{2^2}\right)+\left(\frac{1}{2^2}-\frac{1}{3^2}\right)+\ldots+\left(\frac{1}{14^2}-\frac{1}{15^2}\right)=\left(\frac{1}{1^2}-\frac{1}{15^2}\right)$$

$$=\frac{224}{225}.$$

3.Sol: Rationalise the denominator, we get

$$\sum_{k=1}^{n}\frac{(k+1)\sqrt{k}-k\sqrt{k+1}}{(k+1)^2k-k^2(k+1)}=\sum_{k=1}^{n}\frac{(k+1)\sqrt{k}-k\sqrt{k+1}}{k(k+1)}$$

$$=\sum_{k=1}^{n}\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}$$

The telescope as $\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{n}}$

4.Sol: Factor each term in the product as a difference of two squares, and group together all the terms that contain a $-$ sign, and all those that contain a $+$ sign. This gives

$$\left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)...\left(1-\frac{1}{10}\right)\right]$$

$$\cdot\left[\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)...\left(1+\frac{1}{10}\right)\right]$$

$$=\left[\frac{1}{2}\frac{2}{3}\frac{3}{4}...\frac{9}{10}\right]\cdot\left[\frac{3}{2}\frac{4}{3}\frac{5}{4}...\frac{11}{10}\right]$$

$$=\frac{1}{10}\cdot\frac{11}{2}=\frac{11}{20}$$

5.Sol: Given series is

$$\prod_{k=2}^{2016}\frac{k^2}{k^2-1}=\prod_{k=2}^{2016}\frac{k}{k-1}\prod_{k=2}^{2016}\frac{k}{k+1}=\frac{2006}{1}\frac{2}{2007}=\frac{4012}{2007}$$

6.Sol: If we factor: $n^3-1=(n-1)(n^2+n+1), n^3+1$

$=(n+1)(n^2-n+1)$, then most of the terms cancel. Take the product up to $n=N$. Then the numerator is

$1\cdot2\cdot3...(N-1)\cdot7\cdot13\cdot21\cdot31...(N^2+N+1)$

and the denominator is

$3\cdot4\cdot5...(N+1).3\cdot7\cdot13\cdot21\cdot31...(N^2-N+1)$.

Hence the product up to $n=N$ is

$$\frac{2}{(N(N+1))}\cdot\frac{(N^2+N+1)}{3}=\frac{2}{3}\frac{(N^2+N+1)}{(N(N+1))},$$

which tends to $\frac{2}{3}$ as N tends to infinity.

7.Sol: We use the identity $\frac{\sin 2\theta}{\sin\theta}=2\cos\theta$. Then we get

$$2\cos\theta=\frac{\sin 2\theta}{\sin\theta}$$

$$2\cos 2\theta=\frac{\sin 4\theta}{\sin 2\theta}$$

$$2\cos 4\theta=\frac{\sin 8\theta}{\sin 4\theta}$$

$$2\cos 8\theta=\frac{\sin 16\theta}{\sin 8\theta}$$

$$2\cos 16\theta=\frac{\sin 32\theta}{\sin 16\theta}.$$

Thus, the product can be written as $\frac{\sin 32\theta}{32\sin\theta}$.

8.Sol: Evaluating the first several partial sums of this series, we get the sequence $\frac{1}{2},\frac{2}{3},\frac{3}{4}$, and soon, which suggests the answer is $\frac{99}{100}$. Indeed, this cancellation is not coincidental: if one writes

$\frac{1}{2}=1-\frac{1}{2},\frac{1}{6}=\frac{1}{2}-\frac{1}{3},\frac{1}{12}=\frac{1}{3}-\frac{1}{4}$, and so on, then the sum becomes

$$\left(1-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+...+\left(\cancel{\frac{1}{99}}-\frac{1}{100}\right)$$

$$=1-\frac{1}{100}=\frac{99}{100}.$$

Here is an example of a more difficult telescoping summation.

9.Sol: Note that

$$(k^2+1)k!=k(k\cdot k!)+k!=k[(k+1)!-k!]+k!$$

$$=k(k+1)!-k\cdot k!+k!=k(k+1)!-k!(k-1).$$

Hence for all positive integer k we have

$b_k=a_1+a_2+a_3+...+a_k$

$=1\cdot2!-1!\cdot0+2\cdot3!+3\cdot4!-3!\cdot2+$

$...+k(k+1)!-k!(k-1)$

$$= k(k+1)!$$

Thus $\dfrac{a_k}{b_k} = \dfrac{(k^2+1)k!}{k(k+!)!} = \dfrac{k^2+1}{k(k+1)} = \dfrac{k^2+1}{k^2+k}$.

Substituting $k = 100$ gives $\dfrac{a_{100}}{b_{100}} = \dfrac{10001}{10100}$, and the requested answer is $10100 - 10001 = 99$.

10.Sol: we have

$$\sum_{r=1}^{44} \frac{1}{\cos r^0 \cos(r+1)^0} = \frac{1}{\sin 1^0}\left[\tan 45^0 - \tan 1^0\right]$$

$$= \frac{1}{\sin 1^0}\left[\frac{\sin 44^0}{\cos 1^0 \cos 45^0}\right]$$

$$= 2\sqrt{2}\frac{\sin 44^0}{\sin 2^0}$$

11.Sol: Given sum can be written as

$$S_n = \sum_{r=1}^{n} \frac{\sin 2^{r-1}\theta}{\cos 2^{r-1}\cos 2^r\theta} = \sum_{r=1}^{n} \frac{\sin\left(2^r\theta - 2^{r-1}\theta\right)}{\cos 2^{r-1}\theta \cos 2^r\theta}$$

$$= \sum_{r=1}^{n}\left[\tan 2^r\theta - \tan 2^{r-1}\theta\right] = \tan 2^n\theta - \tan\theta$$

12.Sol: We know that $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$$\Rightarrow \tan A \tan B = \frac{\tan A - \tan B}{\tan(A-B)} - 1 \qquad (1)$$

Let $S_n = \sum_{r=1}^{n} \tan rx \tan(r+1)x$

Then $S_n = \sum_{r=1}^{n}\left[\dfrac{\tan(r+1)x - \tan rx}{\tan x} - 1\right]$

(from (1))

$$= \frac{1}{\tan x}\sum_{r=1}^{n}\left[\tan(r+1)x - \tan rx\right] - \sum_{r=1}^{n} 1$$

$$= \frac{1}{\tan x}\left[\tan(n+1)x - \tan x\right] - n$$

$$= \frac{\tan(n+1)x}{\tan x} - (n+1)$$

13.Sol: $\sum_{r=1}^{88} \tan r^0 \tan(r+1)^0 = \dfrac{\tan 89^0}{\tan 1^0} - 89$

$$= \cot^2 1^0 - 89$$

14.Sol: We have $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$;

$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta);$$

$$\sin^3\frac{\theta}{3} = \frac{1}{4}\left(3\sin\frac{\theta}{3} - \sin\theta\right);$$

$$3\sin^3\frac{\theta}{3^2} = \frac{1}{4}\left(3^2\sin\frac{\theta}{3^2} - \sin\frac{\theta}{3}\right);$$

$$3^2\sin^3\frac{\theta}{3^3} = \frac{1}{4}\left(3^2\sin\frac{\theta}{3^3} - 3^2\sin\frac{\theta}{3^2}\right);$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$3^{n-1}\sin^3\frac{\theta}{3^n} = \frac{1}{4}\left[3^n\sin\frac{\theta}{3^n} - 3^{n-1}\sin\frac{\theta}{3^{n-1}}\right]$$

Hence by adding the required sum

$$\frac{1}{4}\left[3^n\sin\frac{\theta}{3^n} - \sin\theta\right]$$

15.Sol: $S_n = \dfrac{\sin\theta}{\sin 2\theta} + \dfrac{\sin\theta}{\sin 2^2\theta} + \dfrac{\sin\theta}{\sin 2^3\theta} + \ldots + \dfrac{\sin\theta}{\sin 2^n\theta}$

$$= \sin\theta\left(\cot\theta - \cot 2^n\theta\right)$$

MOCK TEST PAPER

JEE MAIN - 7

2018 2019

1. Range of $\{(1,x),(1,y),(2,x),(2,y),(3,z)\}$ is
(a) $\{1,2,3\}$ (b) $\{x,y,z\}$
(c) $\{1,x\}$ (d) $\{1,2,3,x,y,z\}$

2. On the set N of all natural numbers, define the relation R by aRb, if GCD of a and b is 2. Then, R is
(a) Reflexive, but not symmetric
(b) Symmetric only
(c) Reflexive and transitive
(d) Reflexive, transitive and symmetric

3. The function f is continuous for $-2 \le x \le 1$ and $f(-2) = f(2) = 0$. If there is no c, where $-2 < c < 2$, for which $f'(c) = 0$, which of the following statements must be true ?
(a) For $-2 < k < 2,\ f'(k) < 0$
(b) For $-2 < k < 2,\ f(k)$ exists
(c) For $-2 < k < 2,\ f'(k)$ exists, but f' is not continuous
(d) For some k, where $-2 < k < 2,\ f'(k)$ does not exist

4. The maximum and minimum value of $f(x) = |2x+1| - 2|x-1| - 3$ are
(a) 0,-1 (b) 1,-1 (c) -1,-3 (d) 0,-6

5. If p and q are positive real numbers such that $p^2 + q^2 = 1$, then the maximum value $p+q$, is
(a) 2 (b) $\sqrt{2}$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

6. $\int x^2 \cos\frac{x}{2}\,dx$
(a) $2x^2 \sin\frac{x}{2} - 8x\cos\frac{x}{2} - 16\sin\frac{x}{2} + c$
(b) $2x^2 \sin\frac{x}{2} + 8x\cos\frac{x}{2} + 16\sin\frac{x}{2} + c$
(c) $2x^2 \sin\frac{x}{2} - 8x\cos\frac{x}{2} + 16\sin\frac{x}{2} + c$
(d) $2x^2 \sin\frac{x}{2} + 8x\cos\frac{x}{2} - 16\sin\frac{x}{2} + c$

7. The value of the integral $\int_0^{\pi/2} \frac{dx}{2+\cos x}$ can be expressed in the form $\frac{\pi\sqrt[A]{B}}{C}$ where A, B, C are positive integers and B is not divisible by the square of any prime. Find the value of $A+B+C$.
(a) 14 (b) 12 (c) 11 (d) 15

8. Area bounded by the curves $y = \left[\frac{x^2}{64} + 2\right], \forall x \in (-8,8)$ $y = x-1$ and $x = 0$ and above x-axis ($[\cdot]$ denotes the greatest integer function)
(a) 4 (b) 3
(c) 2 (d) None of these

9. If points (5, 5), (10, k) and (-5, 1) are collinear, then $k =$
(a) 3 (b) 5 (c) 7 (d) 9

10. A ray of light through (2, 1) is reflected at a point A on the y-axis and then passes through the point (5, 3). Then co-ordinates of A are :

(a) $\left(0,\frac{11}{7}\right)$ (b) $\left(0,\frac{5}{11}\right)$ (c) $\left(0,\frac{11}{5}\right)$ (d) $\left(0,\frac{3}{5}\right)$

11. The length of the common chord of the circles $x^2+y^2+2x+3y+1=0$ and $x^2+y^2+4x+3y+2=0$, is

(a) $\frac{9}{2}$ (b) $2\sqrt{2}$ (c) $3\sqrt{2}$ (d) $\frac{3}{2}$

12. Locus of the point of intersection of straight lines $\frac{x}{a}-\frac{y}{b}=m$ and $\frac{x}{a}+\frac{y}{b}=\frac{1}{m}$ is

(a) An ellipse (b) A circle
(c) A hyperbola (d) A parabola

13. Number of intersecting points of this conic $4x^2+9y^2=1$ and $4x^2+y^2=4$ is

(a) 1 (b) 2 (c) 3 (d) 0 (zero)

14. The sum of three consecutive terms of an A.P. is 15 and the sum of their squares is 83. find the terms.

(a) 3, 6, 9 (b) 2, 4, 6 (c) 3, 5, 7 (d) 2, 7, 10

15. If $|z+1|=\sqrt{2}|z-1|$, then the locus described by the point z in the Argand diagram is a

(a) Straight line (b) Circle
(c) Parabola (d) None of these

16. The coefficient of x^5 in the expansion of $\left(2-x+3x^2\right)^6$ is

(a) –4692 (b) –4694 (c) –4682 (d) 4592

17. If the probability of A to fail in an examination is 0.2 and that for B is 0.3, then probability that either A or B is fail, is

(a) 0.5 (b) 0.44 (c) 0.8 (d) 0.25

18. The vector equation of a plane, which is at a distance of 8 unit from the origin and which is normal to the vector $2i+j+2k$, is

(a) $r\cdot(2i+j+k)=24$ (b) $r\cdot(2i+j+2k)=24$
(c) $r\cdot(i+j+k)=24$ (d) None of these

19. Solution of equation $\left(xy^4+y\right)dx-xdy=0$ is

(a) $4x^4y^3+3x^3=cy^3$ (b) $3x^3y^4+4y^3=cx^3$
(c) $3x^4y^3+4x^3=cy^3$ (d) None of these

20. The point of intersection of lines $\frac{x-4}{5}=\frac{y-1}{2}=\frac{z}{1}$ and $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ is

(a) $(-1,-1,-1)$ (b) $(-1,-1,1)$
(c) $(1,-1,-1)$ (d) $(-1,1,-1)$

21. The number of solution of the equation $\sqrt{3}\sin x+\cos x=1$ in the interval $0\le x\le 2\pi$.

(a) 3 (b) 2 (c) 4 (d) None of these

22. If $x=a(t-\sin t)$ and $y=a(1-\cos t)$, then $\frac{dy}{dx}=$

(a) $\tan\left(\frac{t}{2}\right)$ (b) $-\tan\left(\frac{t}{2}\right)$
(c) $\cot\left(\frac{t}{2}\right)$ (d) $-\cot\left(\frac{t}{2}\right)$

23. The value of $\lim_{x\to\infty}\frac{x^2+x+4}{x^2+ax+5}$ is

(a) $\frac{b}{a}$ (b) 1 (c) 0 (d) $\frac{4}{5}$

24. Two non-zero distinct numbers a,b are used as elements to make determinants of the third order. The number of determinants whose value is zero for all a,b is

(a) $a+b$ (b) 24
(c) 32 (d) None of these

25. After applying $R_2\to R_2-2R_1$ to $C=\begin{bmatrix}1&2\\2&-1\end{bmatrix}$, we get

(a) $\begin{bmatrix}1&4\\2&-3\end{bmatrix}$ (b) $\begin{bmatrix}1&2\\0&-5\end{bmatrix}$
(c) $\begin{bmatrix}2&-1\\1&2\end{bmatrix}$ (d) $\begin{bmatrix}1&2\\2&-5\end{bmatrix}$

26. The horizontal distance between two towers is 60 m and the angle of depression of the top of the first tower as seen from the top of the second is $30°$. If the height of the second tower is 150 m, then the height of the first tower is

(a) $90m$ (b) $\left(150-60\sqrt{3}\right)m$

(c) $\left(150+20\sqrt{3}\right)m$ (d) None of these

27. The set of all solutions of the inequation $x^2-2x+5\le 0$ is

(a) $R-(-\infty,-5)$ (b) $R-(-\infty,-6)$

(c) ϕ (d) $R-(-\infty,-4)$

28. In a box there are 10 balls ; 4 red, 3 black, 2 white and 1 yellow. In how many ways can a child select 4 balls out of these 10 balls ? (Assume that the balls of the same colour are identical)

(a) 20 (b) 18 (c) 19 (d) 17

29. In a binomial distribution the probability of getting a success is 1/4 and standard deviation is 3, then its mean is

(a) 6 (b) 8 (c) 12 (d) 10

30. The negation of the statement "2 + 3 = 5 and 8 < 10" is-

(a) $2+3\ne 5$ and $8\nless 10$ (b) $2+3\ne 5$ or $8>10$

(c) $2+3\ne 5$ or $8\ge 10$ (d) None of these

ANSWER KEY

1. b	**2.** b	**3.** d	**4.** d	**5.** b
6. d	**7.** a	**8.** a	**9.** c	**10.** a
11. b	**12.** c	**13.** d	**14.** c	**15.** b
16. a	**17.** b	**18.** b	**19.** c	**20.** a
21. a	**22.** c	**23.** b	**24.** c	**25.** b
26. c	**27.** c	**28.** a	**29.** c	**30.** c

HINTS & SOLUTIONS

2.Sol: aRa, then GCD of a and a is a.

$\therefore$ R is not reflexive

Now, $aRb \Leftrightarrow bRa$

If GCD of a and b is 2, then GCD of b and a is 2.

$\therefore$ R is symmetric.

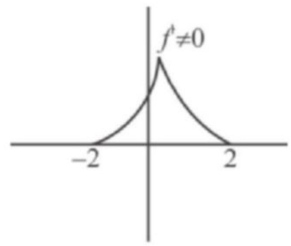
Now $aRb, bRc \Rightarrow aRc$

If GCD of a and b is 2 and GCD of b and c is 2, then it need not be GCD of a and c is 2.

$\therefore$ R is not transitive.

e.g., $6R2, 2R12$ but 6R2

3.Sol: Using extreme value theorem, we can say there is a point k, where critical point exist. So $f'(k)$ does not exist.



4.Sol: Given function is rewritten as

$$f(x)=\begin{cases} 0; & x\ge 1 \\ 4x-4; & \frac{-1}{2}\le 1\ x<1 \\ -6; & x<\frac{-1}{2}\end{cases}$$

$\therefore$ Maximum value is '0' and minimum value is '-6'.

5.Sol: Given $p^2+q^2=1$. so, $q=\sqrt{1-p^2}$

Then $p+q=p+\sqrt{1-p^2}$

$$\Rightarrow f'(p)=1+\frac{1}{2}(1-p^2)^{-1/2}\times(-2p)$$

$\Rightarrow f'(p)=0 \Rightarrow p=\frac{1}{\sqrt{2}}$ (as it is given that p and q are positive)

Also, $f''(p)=-\frac{1}{(1-p^2)^{3/2}}$

Clearly $f''(p)<0$ for $p=\frac{1}{\sqrt{2}}$

$\Rightarrow f(p)$ is maximum when $p=\frac{1}{\sqrt{2}}$

$\Rightarrow p+q$ is maximum when $p=\frac{1}{\sqrt{2}}$

For $p=\frac{1}{\sqrt{2}}$, we get $q=\frac{1}{\sqrt{2}}$

$\therefore p+q=\sqrt{2}$

6.Sol: $\int x^2\cos\frac{x}{2}dx = 2x^2\sin\frac{x}{2} - 4\int x\sin\frac{x}{2}dx$

$$= 2x^2\sin\frac{x}{2} - 4\int -2x\cos\left(\frac{x}{2}\right) + 2\int\cos\frac{x}{2}dx$$

$$= 2x^2\sin\frac{x}{2} - 4\left(-2x\cos\frac{x}{2} + 4\sin\frac{x}{2}\right) + c$$

$$= 2x^2\sin\frac{x}{2} + 8x\cos\frac{x}{2} - 16\sin\frac{x}{2} + c$$

7.Sol: The method that works best is Weierstrass substitution. It starts by dening $t = \tan(x/2)$. Then we have

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

Further more we have

$$\frac{dt}{dx} = \frac{1}{2}\sec^2(x/2) = \frac{1+t^2}{2} \Rightarrow dx = \frac{2dt}{1+t^2}$$

Subbing all these and simplifying yields

$$\int\frac{dx}{2+\cos x} = \int\frac{2dt}{3+t^2}$$

Now let $t = \sqrt{3}\tan z$ so that LaTex :

$dt = \sqrt{3}\sec^2 z dz$. Subbing these gives

$$\int\frac{2dt}{3+t^2} = 2\int\frac{\sqrt{3}\sec^2 dz}{3+3\tan^2 z} = 2\int\frac{\sqrt{3}\sec^2 zdz}{3\sec^2 z} = 2\int\frac{dz}{\sqrt{3}} = \frac{2z}{\sqrt{3}} + C.$$

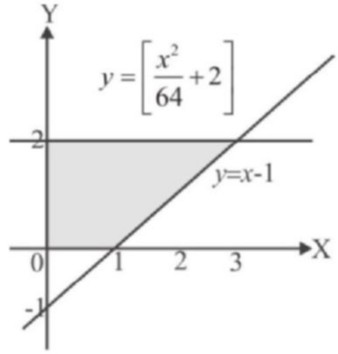
Now subbing back

$$z = \arctan\left(\frac{t}{\sqrt{3}}\right) = \arctan\left[\frac{\tan(x/2)}{\sqrt{3}}\right] \text{ gives}$$

$$\int_0^{\frac{\pi}{2}}\frac{dx}{2+\cos x} = \frac{2}{\sqrt{3}}\arctan\left[\frac{\tan(x/2)}{\sqrt{3}}\right]_0^{\frac{\pi}{2}} + C.$$

So plugging in the values we get $\pi\sqrt{3}/9$. Hence the answer is 14.

8.Sol: $-8 < x < 8 \Rightarrow y = 2$



$\therefore$ Required area is $\frac{1}{2}(1+3)\times 2 = 4$ squnits

9.Sol: Given points are colinear.

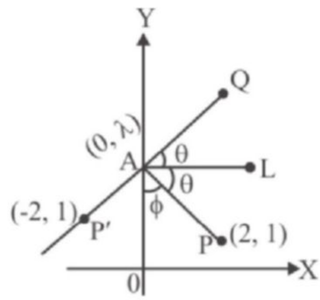
i.e., $$\begin{vmatrix} 5 & 5 & 1 \\ 10 & k & 1 \\ -5 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 5 & 5 & 1 \\ 5 & k-5 & 0 \\ -10 & 1-5 & 0 \end{vmatrix} = 0 \Rightarrow k = 7$$

10.Sol: Let the coordinates of the point be $A(0,\lambda)$.

Let AL be the perpendicular to y-axis.

By the law of reflection, we know angle of incidence = angle of reflection



i.e., $\angle PAL = \angle LAQ = \theta$

$$\angle YAQ = 180° - (\phi + 2\theta)$$

$$= 180° - (\phi + 2(90° - \phi))$$

$$\therefore \angle YAQ = 180° - \phi - 180 + 2\phi$$

$$= \phi$$

$\Rightarrow \qquad \angle QAP = 180° - \phi$

Slope of the line A.P, $m = \dfrac{1-\lambda}{2}$ also $m = \tan\theta$

$\Rightarrow \qquad \tan\phi = \dfrac{1-\lambda}{2}$

(1) slope of the line $AQ = \dfrac{3-\lambda}{5}$

$\therefore \tan(180° - \phi) = \dfrac{3-\lambda}{5}$

$\Rightarrow \qquad -\tan\phi = \dfrac{3-\lambda}{5}$

(2) equating (1) and (2), we get

$$\frac{1-\lambda}{2} = \frac{\lambda-3}{5}$$

$\Rightarrow \qquad 5 - 5\lambda = 2\lambda - 6$

$7\lambda = 11$

$$\lambda = \frac{11}{7}$$

$\therefore$ A is $\left(0, \dfrac{11}{7}\right)$

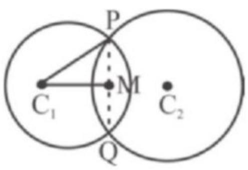
11.Sol: Equation of common chord is

$x^2 + y^2 + 2k + 3y + 1 - x^2 - y^2 - 4x - 3y - 2 = 0$

i.e., $\qquad -2x - 1 = 0$

$\Rightarrow \qquad 2x + 1 = 0$

$\therefore$ The equation of common chord PQ is $2x + 1 = 0$



we know $PQ = 2PM$, but $PM = \sqrt{c_1p^2 - c_1m^2}$

$\therefore \qquad C_1M = \dfrac{|2(-1)+1|}{\sqrt{2^2}} = \dfrac{1}{2}$

now $\qquad PM = \sqrt{\dfrac{9}{4} - \dfrac{1}{4}} = \sqrt{\dfrac{8}{4}} = \sqrt{2}$

$\therefore$ Length of common chord is $2PM = 2\sqrt{2}$

12.Sol: Given that $\dfrac{x}{a} - \dfrac{y}{b} = m \qquad (1)$

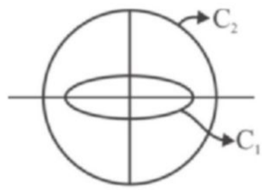
$$\frac{x}{a} + \frac{y}{b} = \frac{1}{m} \qquad (2)$$

Multiplying equation (1) and (2), we get

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = m \cdot \frac{1}{m}$$

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, which is the equation of hyperbola.

13.Sol: $C_1 : \dfrac{x^2}{1/4} + \dfrac{y^2}{1/9} = 1 \quad C_2 : \dfrac{x^2}{1} + \dfrac{y^2}{4} = 1$



Clearly no intersection.

14.Sol: Three consecutive terms of an A.P. are taken as:

$$(a-d), a, (a+d)$$

$\Rightarrow \qquad (a-d) + a + (a+d) = 15 \qquad (1)$

$$(a-d)^2 + a^2 + (a+d)^2 = 83 \qquad (2)$$

From eqn (1), we get: $a = 5$.

Substituting for $a = 5$ in eqn (2), we get:

$d^2 = 4. \qquad$ i.e., $d = \pm 2$

Hence the terms are:

3, 5, 7 or 7, 5, 3.

15.Sol: Let $z = x + iy$

Now, $|z+1| = \sqrt{2}|z-1|$

i.e., $|(x+1) + iy| = \sqrt{2}|(x+1) + iy)|$

$\Rightarrow \quad (x+1)^2 + y^2 = 2(x-1)^2 + 2y^2$

$\Rightarrow \quad x^2 + y^2 - 6x + 1 = 0$

$\therefore$ Locus of $|z+1| = \sqrt{2}|z-1|$ is a circle.

16.Sol: We have $\left(2 - x + 3x^2\right)^6 = \left[2 - x(1-3x)\right]^6$

$$=\left[x(1-3x)-2\right]^6$$

Now $\left[x(1-3x)-2\right]^6=\binom{6}{0}x^6(1-3x)^6$

$$-\binom{6}{1}x^5(1-3x)^5\cdot 2$$

$$+\binom{6}{2}x^4(1-3x)^4 2^2+\binom{6}{3}x^3(1-3x)^3 2^3$$

$$+\binom{6}{4}x^2(1-3x)^2\cdot 2^4-\ldots$$

Thus, the coeficient of x^5 in $\left(2-x+3x^2\right)^6$ is

$$-\binom{6}{1}\cdot 2+\binom{6}{2}4\cdot(-3)\cdot 2^2-\binom{6}{3}\cdot 3(-3)^3\cdot 2^3$$

$$=-12-1440-3240=-4692$$

17.Sol: Given that

$$P\left(\overline{A}\right)=0.2\Rightarrow P(A)=1-0.2=0.8$$

and $P\left(\overline{B}\right)=0.3\Rightarrow P(B)=1-0.3=0.7$

now, the required probability is

$$P\left(\overline{A}\cup\overline{B}\right)=P\left(\overline{A\cap B}\right)=1-P(A\cap B)$$

$$=1-P(A)P(B)$$

$$=1-(0.8)(0.7)$$

$$=1-0.56$$

$$=0\cdot 44$$

18.Sol: Here, $d=8$ and $n=(2i+j+2k)$

$$\therefore \hat{n}=\frac{n}{|n|}=\frac{2i+j+2k}{\sqrt{4+1+4}}=\frac{2}{3}i+\frac{1}{3}j+\frac{2}{3}k$$

Hence, the required equation of the plane is

$$r\cdot\left(\frac{2}{3}i+\frac{1}{3}j+\frac{2}{3}k\right)=8 \text{ and } r\cdot(2i+j+2k)=24$$

19.Sol: Rewrite the equation as

$$\frac{dy}{dx}-\frac{y}{x}=y^4$$

This ia a Bernoulli's equation. Divide through by y^4

$$y^{-4}\frac{dy}{dx}-\frac{1}{x}y^{-3}=1$$

make the substitution $t=y^{-3}$ to obtain

$$\frac{dt}{dx}+\frac{3}{x}t=-3$$

This is now just regular linear first order equation.

The integrating factor $e^{3\ln x}=x^3$

$$\Rightarrow \quad x^3\frac{dt}{dx}+3x^2t=-3x^3$$

Integrating on both sides

$$\int d\left(x^3t\right)=-3\int x^3dx$$

$$x^3t=-3\frac{x^4}{4}+c$$

$$\Rightarrow \quad \frac{x^3}{y^3}=-3\frac{x^4}{4}+c$$

i.e., $3x^4y^3+4x^3=cy^3$

77. c

20.Sol: Let the equation of the line

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=t$$

$\therefore$ coordinates of the point on the line is given as

$x=2t+1, y=3t+2$ and $z=4t+3$ (1)

likewise, for the equation

$$\frac{x-4}{5}=\frac{y-1}{2}=\frac{z-0}{1}=s$$

the coordinates of any point on this line is given as

$$x=5s+4, y=2s+1, z=s \quad (2)$$

we know, if the lines intersects, the point of intersection will lies on both equations.

i.e., $2t+1=5s+4$

$\Rightarrow$ $2t-5s=3$

$\Rightarrow$ $3t+2=2s+1$

$\Rightarrow$ $3t-2s=-1$

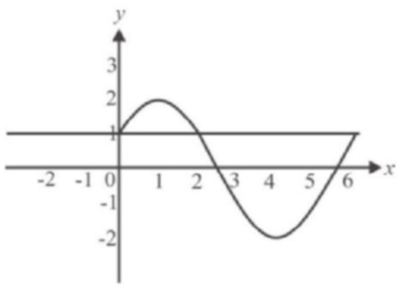
Solving (3) and (4), we get $s=-1$ and $t=-1$

Now the z-coordinate of eq (1) is

$4(-1)+3=4+3=-1,$ which is equal to z-coordinate of eq(2).

$\therefore$ The intersecting point is $(-1,-1,-1)$

21.Sol:



From the graph it is clear that, it has 3 solutions

Aliter:

Given that $\sqrt{3}\sin x + \cos x = 1$

$$\cos\left(x - \frac{\pi}{3}\right) = \frac{1}{2}$$

i.e., $$x - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{3}$$

$$\Rightarrow \quad x = 2n\pi + 2\frac{\pi}{3} \text{ or } x = 2n\pi$$

To get particular solution satisfying $0 \le x \le 2\pi$, we will substitute integral values of n

i.e., $$n = 0 \Rightarrow x = 0, 2\frac{\pi}{3}$$

$$n = 1 = x = 2\pi$$

22.Sol: Given that $x = a(t - \sin t)$ and $y = a(1 - \cos t)$

now, $\frac{dy}{dx} = a(\sin t)$ and $\frac{dx}{dt} = a(1 - \cos t)$

$$\Rightarrow \frac{dy}{dx} = \frac{a(1-\cos t)}{a \sin t} = \cot\left(\frac{t}{2}\right)$$

23.Sol: $$\lim_{x\to\infty} \frac{x^2 + x + 4}{x^2 + ax + 5} = \lim_{x\to\infty} \frac{1 + \frac{1}{x} + \frac{4}{x^2}}{1 + \frac{a}{x} + \frac{5}{x^2}}$$

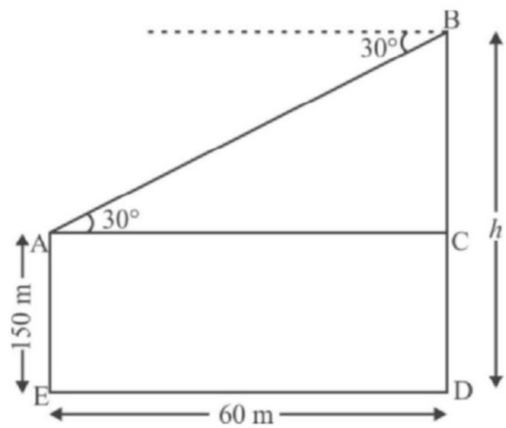
24.Sol: As the determinant has to vanish for all a, b we have atleast two rows or columns are identical. After filling the first column in $2\times2\times2$ ways and filling another column. likewise, the remaining column can be filled in $2\times2\times2$ ways. so, the number of ways is 2^6.

But each of the two ways give the same determinant.

Therefore, the required number of determinants is 32.

25.Sol: Applying $R_2 \to R_2 - 2R_1$ to $C = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, we get $\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$

26.Sol: In ΔABC, $\tan 30^\circ = \frac{BC}{AC}$



$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{h - 150}{60}$$

$$\Rightarrow h\sqrt{3} - 150\sqrt{3} = 60$$

$$\therefore \quad h = \frac{60 + 150\sqrt{3}}{\sqrt{3}} = \left(150 + 20\sqrt{3}\right)m$$

27.Sol: Given that $x^2 - 2x + 5 \le 0$

$$\Rightarrow \quad (x-1)^2 + 4 \le 0$$

$$\therefore \quad x = \phi.$$

28.Sol: Let x_1, x_2, x_3 and x_4 be the number of red, black, white, yellow balls selected respectively.

Number of ways to select 4 balls = Number of integral solution of the equation

$$x_1 + x_2 + x_3 + x_4 = 4$$

Conditions on x_1, x_2, x_3 and x_4

The total number of red, black, white and yellow balls in the box are 4, 3, 2 and 1 respectively.

So we can take : $\text{Max}(x_1)=4, \text{Max}(x_2)=3,$

$\text{Max}(x_3)=2, \text{Max}(x_1)=1$

There is no condition on minimum number of red, black, white and yellow balls selected, so take :

$\text{Min}(x_i)=0$ for $i=1,2,3,4$

Number of ways to select 4 balls

$=$ coeff of x^4 in

$$\left(1+x+x^2+x^3+x^4\right)\times\left(1+x+x^2+x^3\right)\times\left(1+x+x^2\right)\times(1+x)$$

$=$ coeff of x^4 in

$$\left(1-x^5\right)\left(1-x^4\right)\left(1-x^3\right)\left(1-x^2\right)(1-x)^{-4}$$

$=$ coeff of x^4 in $(1-x)^{-4}$ $-$ coeff. of x^2 in $(1-x)^{-4}$ $-$ coeff of x^1 in $(1-x)^{-4}$ $-$ coeff. of x^0 in $(1-x)^{-4}$

$$= {}^7C_4 - {}^5C_2 - {}^4C_1 - {}^3C_0$$

$$= \frac{7\times6\times5}{3!} = 10-4-1 = 35-15=20$$

Thus, number of ways of selecting 4 balls from the box subjected to the given conditions is 20.

29.Sol: Probability of success $p=\frac{1}{4}$

Probability failure $q=\frac{3}{4}$

we have mean $=np,$ and

Standard deviation $=\sqrt{\text{Variance}} \Rightarrow \text{Variance}=9$

i.e., $npq=9 \Rightarrow n.\frac{1}{4}.\frac{3}{4}=9 \Rightarrow n=48$

$\text{Mean}=np=\frac{1}{4}\times48=12$

QUICK RECAP

For competitive edge

MATRICES

(1) Formal Definition

A matrix is a table of m lines and n columns

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

It is denotes by $A_{m\times n}$ matrices.

(2) Some Special Matrix

Definition (principal diagonal): The principal diagonal is made of elements of the form a_{ii} of a square matrix $a_{11}, a_{22}, a_{33}, \ldots a_{nn}$.

Definition: Let $A=\left[a_{ij}\right]_{m\times n}$ be a square matrix.

(i) If $a_{ij}=0$ for all i, j, then A is called a zero matrix.

(ii) If $a_{ij}=0$ for all $i<j$, then A is called a lower triangular matrix.

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{12} & a_{22} & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix}$$

Lower triangular matrix

Definition: Let $A=\left[a_{ij}\right]_{m\times n}$ be a square matrix.

If $a_{ij}=0$ for all $i\neq j$, then A is called a diagonal matrix.

Definition: If A is diagonal matrix and $a_{11}=a_{22}=\cdots=a_{nn}=m$, where m is any real number, then A is called a scalar matrix.

Definition: If A is a diagonal matrix and $a_{11}=a_{22}=\cdots=a_{nn}=1$, then A is called an identity matrix or a unit matrix,

(3) Orthogonal Matrix

Any square matrix A of order n is said to be orthogonal, if $AA'=A'A=I_n$.

(4) Idempotent Matrix

A square matrix A is called idempotent provided it satisfies the relation $A^2=A$.

(5) Involuntary Matrix

A square matrix such that $A^2=I$ is called involuntary matrix.

(6) Nilpotent matrix

A square matrix A is called a nilponent matrix if there exist a positive integer m such that $A^m=O$.

If m is the least positive integer such that $A^m=O$, then m is called the index of the nilpotent matrix A.

(7) Properties of Matrix Addition

Theorem: Let A,B,C be matrices of the same order and O be the zero matrix of the same order. Then

(i) $A+B=B+A$

(ii) $(A+B)+C=A+(B+C)$

(iii) $A+(-A)=(-A)+A=O$

(iv) $A+O=O+A$

(8) Properties of Matrix Multiplication

Theorem:

(i) $(AB)C=A(BC)$

(ii) $A(B+C)=AB+AC$

(iii) $(A+B)C=AC+BC$

(iv) $AO=OA=O$

(v) $IA=AI=A$

(vi) $k(AB)=(kA)B=A(kB)$

(vii) $(AB)^T=B^T\cdot A^T$

(9) Properties of scalar multiplication

Theorem: Let A, B be matrices of the same order and h,k be two scalars. Then

(i) $k(A+B)=kA+kB$

(ii) $(k+h)A=kA+hA$

(iii) $(hk)A=h(kA)=k(hA)$

(10) Properties of transpose

Theorem: Let A, B be two mxn matrices and k be a scalar, then

(i) $(A^T)^T=A$

(ii) $(A+B)^T=A^T+B^T$

(iii) $(kA)^T=k\cdot A^T$

(iv) $(A\cdot B)^T=B^T\cdot A^T$

(11) Symmetric

Definition: A square matrix A is called a symmetric matrix if $A^T=A$.

i.e, A symmetric matrix

$\Leftrightarrow A^T=A\Leftrightarrow a_{ij}=a_{ji}\ ,\forall i,j$

Definition: A square matrix A is called a skew-symmetric matrix if $A^T=-A$.

i.e, A is skew-symmetric matrix

$\Leftrightarrow A^T=-A\Leftrightarrow a_{ij}=-a_{ji}\ ,\forall i,j$

(12) Power of Matrices

Definition : For any square matrix A and any positive integer n, the symbol A^n denotes $\underbrace{A\cdot A\cdot A\cdots A}_{n\text{ factors}}$

Theorem of power of matrices:

(i) Let A be square matrix, then $(A^n)^T=(A^T)^n$.

(ii) If $AB=BA$

(I) $(A+B)^n=A^n+C_1^nA^{n-1}B+C_2^nA^{n-2}B^2+\ldots$

$+C_{n-1}^nA^1B^{n-1}+C_n^nA^{n-n}B^n$

(II) $(AB)^n=A^nB^n$

(III) $(A+I)^n=A^n+C_1^nA^{n-1}+C_2^nA^{n-2}+\ldots$

$+C_{n-1}^nA^1+C_n^nI$

(13) Inverse Matrix

Definition: A square matrix A of order n is said to be non-singular or invertible if and only if there exists a square matrix B such that $AB=BA=I$.

The matrix B is called the multiplicative inverse of A, denoted by A^{-1}.

i.e., $AA^{-1}=A^{-1}A=I$

Theorem: The inverse of a non - singular matrix is unique.

Theorem (Properties of Inverse): Let A, B be two non-singular matrices of the same order and be a scalar.

(i) $(A^{-1})^{-1}=A$

(ii) A^T is a non-singular and $(A^T)^{-1}=(A^{-1})^T$

(iii) A^n is a non-singular and $(A^n)^{-1}=(A^{-1})^n$

(iv) λA is a non-singular and $(\lambda A)^{-1}=\dfrac{1}{\lambda}A^{-1}$

(v) AB is a non-singular and $(AB)^{-1}=B^{-1}A^{-1}$

(14) Diagonal Matrix: The elements of a square matrix A for which $i=j$, i.e, $a_{11},a_{22},a_{33},\ldots a_m$ are called diagonal elements and the line joining these elements is called the principal diagonal or leading diagonal of matrix A.

(15) Trace of a Matrix : The sum of diagonal elements of a square matrix. A is called the trace of matrix A, which is denoted by trA.

$$trA = \sum_{i=1}^{n} a_n = a_{11} + a_{22} + \ldots a_m$$

Properties of trace of a matrix

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ and λ be a scalar

- $tr(\lambda A) = \lambda tr(A)$
- $tr(A - B) = tr(A) - tr(B)$
- $tr(AB) = tr(BA)$
- $tr(A) = tr(A')$ or $tr(A^T)$
- $tr(I_m) = m$
- $tr(0) = 0$
- $tr(AB) \neq trA\, trB$

DETERMINANTS

Properties of Determinants

(1) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

i.e., $\det(A^T) = \det A$

(2) If you change two rows (columns) of a matrix you reverse the sign of its determinant from positive to negative or from negative to positive.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix}$$

$$= -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

(3) If A has a row (column) that is called zeros, then $\det A = 0$.

$$A = \begin{vmatrix} a_1 & 0 & c_1 \\ a_2 & 0 & c_2 \\ a_3 & 0 & c_3 \end{vmatrix} = 0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 0 \end{vmatrix}$$

(4) If two rows (columns) of a matrix are equal, its determinant is zero.

$$\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(5) $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$, then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

(6) The determinant behaves like a linear function on the rows (columns)

$$\begin{vmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix}$$

(7) If we multiply one row of a matrix by p, the determinant is multiplied by p.

$$\begin{vmatrix} pa_1 & b_1 & c_1 \\ pa_2 & b_2 & c_2 \\ pa_3 & b_3 & c_3 \end{vmatrix} = p\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ pa_2 & pb_2 & pc_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and $\begin{vmatrix} pa_1 & pb_1 & pc_1 \\ pa_2 & pb_2 & pc_2 \\ pa_3 & pb_3 & pc_3 \end{vmatrix} = p^3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(8) The determinant of a triangular matrix is the product of the diagonal entries $d_1, d_2, \ldots, d_n$.

(9) The determinant of a permutation matrix A is 1 or depending on whether A exchanges an even or odd number of rows (columns).

(10) $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\lambda b_1 & b_1 & c_1 \\ a_2+\lambda b_2 & b_2 & c_2 \\ a_3+\lambda b_3 & b_3 & c_3 \end{vmatrix}$$

(11) If any line of a determinant Δ be passed over p parallel lines, the resultant determinant is $(-1)^p \Delta$.

(12) When the elements of a determinant Δ are rational integral functions of x (polynomials) and two rows or columns become identical when $x = a$, then $(x-a)$ is a factor of Δ. If r rows become identical when a is substituted for x, then $(x-a)^{r-1}$ is a factor of Δ.

(13) Differentiation of determinant

$$\Delta = \begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix}$$

where f_r, g_r, h_r are functions of x for $r = 1, 2, 3$.

$$\therefore \frac{d\Delta}{dx} = \begin{vmatrix} f'_1 & g'_1 & h'_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & g_1 & h_1 \\ f'_2 & g'_2 & h'_2 \\ f_3 & g_3 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f'_3 & g'_3 & h'_3 \end{vmatrix}$$

(14) Optimal value of Determinants when Elements are known

If $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$ where

$a_i's \in \{\alpha_1, \alpha_2, ..., \alpha_n\}$, then $|A|_{\max}$ when diagonal elements are $\min\{\alpha_1, \alpha_2, ..., \alpha_n\}$ and non-diagonal elements are $\max\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n\}$ and also $|A|_{\min} = -|A|_{\max}$.

(15) Multiplication of Determinants

Definition:

Let $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, |B| = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$. Then

$$|A||B| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{vmatrix}$$

Theorem

(1) $\det(AB) = (\det A)(\det B)$ i.e., $|AB| = |A||B|$

(2) $|A|(|BC|) = (|AB|)|C|$; N.B ; $A(BC) = (AB)C$

(3) $|A||B| = |B||A|$; N.B; $AB \neq BA$ in general

(4) $|A|(|B| + |C|) = |A||B| + |A||C|$

(16) Inverse of Square Matrix by Determinants

Definition: The cofactor matrix of A is defined as

$$cofA = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

Definition: The adjoint matrix of A is defined as

$$adjA = (cofA)^T = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{21} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

Theorem: For any square matrix A of order *n*.

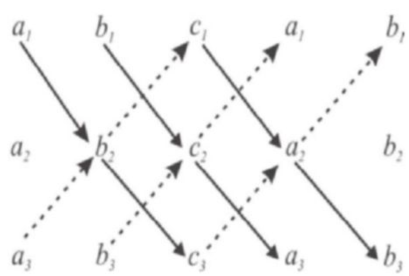
$$A(adj\ A) = (adj\ A)A = (\det A)I$$

$$A(adj\ A) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}$$

Theorem: Let A be a square matrix. If $\det A \neq 0$, then A is non - singular and $A^{-1} = \frac{1}{\det A}(adjA)$.

Theorem: A square matrix A is non-singular iff $\det A \neq 0$.

(17) Sarrus Rule for Expansion



$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3$$

$$-a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

(18) Area of Triangle

The area of a triangle, whose vertices are $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3), is

$$\frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right\|$$

(19) Condition of concurrency of three lines

Three lines are said to be concurrent if they pass through a common point, i.e., they meet at a point.

Let $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$

$a_3x + b_3y + c_3 = 0$; be three concurrent lines,

then $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Condition for General Second Degree Equation in x and y represent pair of straight lines.

The general second degree equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

represents pair of straight lines if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

(20) Cyclic order

$$f(a,b,c) = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$ then

$f(a,a,c) = f(a,b,b) = f(c,b,c) = 0$

then $a-b, b-c, c-a$ are factors of determinant.

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

PRINCIPLES OF MATHEMATICAL INDUCTION

(1) Introduction

Propositions can be divided into general and particular. The following are examples of general propositions:

- All citizens of India have the right to education.
- In every parallelogram the diagonals are bisected at their point of intersection.
- All numbers ending with a zero are divisible by 5.

(2) The corresponding examples of particular propositions are as follows:

- Krishna has the right to education.
- In the parallelogram ABCD the diagonals are bisected at their point of intersection.
- 230 is divisible by 5.

The transition from general proposition to to particular ones is called **Deduction**. Let us consider an example:

(1) All citizens of India have the right to education.
(2) Krishna is an Indian.
(3) Krishna has the right to education.

We obtain the particular proposition (3) from the general proposition (1) , with the help of proposition (2).

Progressing from particular propositions to general ones is called **Induction**. Induction can lead to correct as well as to incorrect conclusions.

(3) The principle of induction

- Is it impossible to prove theorems for infinitely many cases? The opposite is true - usually the

infinite ones are not really interesting because they can be checked case by case e.g. using a computer.

- Mathematical induction is the most fundamental method that allows us to prove theorems for infinitely many cases. The condition is, however, that there are only **countably many** cases, meaning that we can number them.

(4) Definition (Mathematical Induction)

(1) Show the theorem is true for the case having the number 1.

(2) Show that if the theorem is true for some case, then it is is also true for the case with the next number.

THEORY OF EQUATIONS

Quadratic Equation and Inequation

(1) Roots of a Quadratic Equation

Roots of a quadratic equation

$ax^2+bx+c=0 (a \neq 0, a, b, c \in R)$ are given by:

$$\alpha, \beta = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

- Sum of the roots $= \alpha + \beta = -\frac{b}{a}$
- Product of roots $= \alpha\beta = \frac{c}{a}$
- Factorised form of $ax^2+bx+c = a(x-\alpha)(x-\beta)$.
- If S be the sum and P is the product of roots, then quadratic equation is : $x^2 - Sx + P = 0$

(2) Nature of Roots of a Quadratic Equation

(a) If $D<0 (b^2-4ac<0)$, then the roots of the quadratic equation are non – real *i.e.* complex *roots*.

(b) If $D=0 (b^2-4ac=0)$, then the roots are *real* and *equal*.

Equal root $= -\frac{b}{2a}$

(c) If $D>0 (b^2-4ac>0)$, then the roots are *real* and *unequal*.

- If ***D*** *i.e.*, (b^2-4ac) is a perfect square and *a*, *b* and *c* are rational, then the roots are rational.
- If ***D*** *i.e.*, (b^2-4ac) is not a perfect square and *a*, *b* and *c* are *rational*, then roots are of the form $m+\sqrt{n}$ and $m-\sqrt{n}$.
- If $a=1, b, c \in I$ and the roots are rational numbers, then the roots must be *integer*.
- If a quadratic equation in x has more than two roots, then it is an identity in x (*i.e.,* true for all real valuesof x) and $a=b=c=0$.

(3) Condition for Common Root(s)

- **For two common roots:** In such a case, two equations should be identical. For that, the ratio of coefficients of x^2, x and x^0 (i.e., constant) must be same,

  i.e., $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

- **For one common root:** Let α be the common root of two equations. So α should satisfy the two equations.

  $\Rightarrow \quad a\alpha^2 + b\alpha + c = 0$ and $a'\alpha^2 + b'\alpha + c' = 0$

  $\Rightarrow \quad (bc'-b'c)(ab'-a'b) = (a'c-ac')^2$

  This is the condition for one root of two quadratic equations to be common.

(4) Roots of two quadratic equations

- If two quadratic equations $a_1x^2+b_1x+c_1=0$ and $a_2x^2+b_2x+c_2=0$ both have same roots, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
- If the above two quadratic equations have just one root in common then the relation between the coefficients is given by cross multiplication rule

$$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}^2 = \begin{vmatrix} a_1 & b_1 \\ b_2 & b_2 \end{vmatrix} \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

- The quadratic equations $a_1x^2+b_1x+c_1=0$ and $a_2x^2+b_2x+c_2=0$ with Δ_1 and Δ_2 as their discriminants then we have both roots are either real or complex

  (a) $\Delta_1\Delta_2>0 \Rightarrow$ both equation has real roots

  (b) $\Delta_1\Delta_2<0 \Rightarrow$ at least one equation has real roots

  (c) $\Delta_1\Delta_2=0 \Rightarrow$ at least one equation has real roots

  (d) $\Delta_1+\Delta_2=0 \Rightarrow \Delta_1=-\Delta_2 \Rightarrow$ Either exactly one equation has real root or both have identical.

(5) *Inequalities*

The following are some very useful points to remember:

- $a\le b \Rightarrow$ Either $a<b$ or $a=b$
- $a<b$ and $b<c \Rightarrow a<c$
- $a<b \Rightarrow a+c<b+c, \forall c\in R$
- $a<b \Rightarrow -a>-b$ *i.e.*, inequality sign reverses if both sides are multiplied by a negative number.
- $a<b$ and $c<d \Rightarrow a+c<b+d \Rightarrow a-d<b-c$
- $a<b \Rightarrow ma<mb$ if $m>0$ and $ma>mb$ if $m<0$
- $0<a<b \Rightarrow a^r<b^r$ if $r>0$ and $a^r>b^r$ if $r<0$
- $\left(a+\dfrac{1}{a}\right)\ge 2, \forall a>0$ and equality holds for $a=1$.
- $\left(a+\dfrac{1}{a}\right)\le -2, \forall a>0$ and equality holds for $a=-1$.

(6) *Quadratic Inequation*

Let $f(x)=ax^2+bx+c$ where $a,b,c\in R$ and $a\neq 0$. To solve the inequations of type: $\{f(x)\le 0; f(x)<0; f(x)\ge 0; f(x)>0\}$, we use the following procedure.

(a) $D>0$

- Make the coefficient of x^2 positive
- Factorise the expression and represent the left hand side of inequality in the form $(x-\alpha)(x-\beta)$.
- If $(x-\alpha)(x-\beta)>0$, then x lies outside α and β. $(\beta>\alpha)$ all cases

  $\Rightarrow \quad x\in(-\infty,\alpha)\cup(\beta,\infty)$
- If $(x-\alpha)(x-\beta)\ge 0$, then x lies on and outside α and β. $\Rightarrow \quad x\in(-\infty,\alpha]\cup[\beta,\infty)$
- If $(x-\alpha)(x-\beta)<0$, then x lies inside α and β. $\Rightarrow \quad x\in(\alpha,\beta)$
- If $(x-\alpha)(x-\beta)\le 0$, then x lies on and inside α and β. $\Rightarrow \quad x\in[\alpha,\beta]$

(b) $D<0$ and $a>0: f(x)>0$ for all $x\in R$.

(c) $D<0$ and $a<0: f(x)<0$ for all $x\in R$.

(d) $D=0$ and $a>0: f(x)\ge 0$ for all $x\in R$.

(e) $D=0$ and $a<0: f(x)\le 0$ for all $x\in R$.

(f) $D\le 0, a>0: f(x)\ge 0$ for all $x\in R$.

(g) $D\le 0, a<0: f(x)\le 0$ for all $x\in R$.

(7) *Maximum and Minimum values of a Quadratic Polynomial*

Let $f(x)=ax^2+bx+x, a\neq 0$.

Case: I ($a>0$) When $a>0$, parabola opens upward. From graph, vertex (V) is the lowest point on the graph.

$\Rightarrow y=f(x)$ possesses minimum value at

$$x=\frac{-b}{2a}$$

$$\Rightarrow y_{\min}=f(x)_{\min}=\frac{-D}{4a} \text{ at } x=\frac{-b}{2a}$$

Case: II ($a<0$) When $a<0$, parabola opens downward. From graph, vertex (V) is the highest point on the graph.

$y=f(x)$ possesses maximum value at $x=\dfrac{-b}{2a}$

$$\Rightarrow y_{\max}=f(x)_{\max}=\frac{-D}{4a} \text{ at } x=\frac{-b}{2a}$$

(8) *Location of roots*

Theorem: Let $f:[a,b]\to\mathbb{R}$ be a continuous

function such that $f(a) < 0$ and $f(b) > 0$.

Then there is some $c \in (a,b)$ such that $f(c) = 0$.

(i) Condition for a number k to lie between the roots of a quadratic equation

If a number k lies between the roots of a quadratic equation $f(x) = ax^2 + bx + c = 0$, then the equation must have real roots and the sign of $f(k)$ is opposite to the sign of a Combining the above cases, i.e.,

$a > 0, f(k) < 0$ and $a < 0, f(k) > 0$, we get

$af(k) < 0$ for k lie between roots.

(ii) Condition for a number k to be less than roots of a quadratic equation

If a number k is smaller than the roots of a quadratic equation $f(x) = ax^2 + bx + c = 0$, thenthe equation must have real and distinct roots and the sign of $f(k)$ should be the same as the sign of a

Thus, a number k is smaller than the roots of a quadratic equation $ax^2 + bx + c = 0$, if

(i) $\Delta \geq 0$ (ii) $af(k) > 0$

(iii) $k < \dfrac{-b}{2a}$

(iii) Condition for a number k to be greaterthan the roots of a quadratic equation

Thus, a number k is greater than the roots of a quadratic equation $ax^2 + bx + c = 0$, if

(i) $\Delta \geq 0$ (ii) $af(k) > 0$

(iii) $k > \dfrac{-b}{2a}$

(iv) Summary

Case	Condition -I	Condition-II	Condition-III
Number less than roots	$\Delta \geq 0$	af(k) > 0	$k < \frac{-b}{2a}$
Number greater than roots	$\Delta \geq 0$	af(k) > 0	$k > \frac{-b}{2a}$
Number between roots	$\Delta \leq 0$	af(k) < 0	

(v) Condition for both the roots of a quadratic equation to lie between numbers k_1 and k_2.

If both the roots of a quadratic equation lie between numbers k_1 and k_2, then

(i) $\Delta \geq 0$ (ii) $af(k_1) > 0, af(k_2) > 0$

(iii) $k_1 < \dfrac{-b}{2a} < k_2$

(vi) Condition for exactly one of a quadratic equation to lie in the interval (k_1, k_2).

Note: In this case, since exactly one root lies between k_1 and k_2, thus the graph of the quadratic polynomial will cut the x-axis exactly once between k_1 and k_2. This implies that the signs of $f(k_1)$ and $f(k_2)$ would be different. Hence $f(k_1)f(k_2) < 0$.

Thus, exactly one root of the equation $ax^2 + bx + c = 0$ lies in the interval (k_1, k_2) if $f(k_1)f(k_2) < 0$.

(vii) Conditions based on the sum and product of roots of Quadratic equation

- Both roots of $f(x) = ax^2 + bx + c = 0$ are negative if:

  (a) Sum of roots are negative, i.e., $-\dfrac{b}{a} < 0$

  (b) Product of roots are positive, i.e., $\dfrac{c}{a} > 0$

  (c) Discriminant is greater than or equal to Zero i.e., $\Delta \geq 0$.

- Both roots of $f(x) = ax^2 + bx + c = 0$ are Positive if:

  (a) Sum of roots are negative, i.e., $-\dfrac{b}{a} > 0$

  (b) Product of roots are positive, i.e., $\dfrac{c}{a} > 0$

  (c) Discriminant is greater than or equal to Zero i.e., $\Delta \geq 0$.

- Both roots of $f(x) = ax^2 + bx + c = 0$ are of opposite sign if :

  (a) Product of roots are negative, i.e., $\dfrac{c}{a} < 0$

  (b) Discriminant is greater than or equal to Zero, i.e., $\Delta \geq 0$.

COMPLEX NUMBERS

(1) Basics

Definition: A complex number is a number of the format : $z = x + iy, x, y \in R$, where $i^2 = -1$, x is a real part, iy is an imaginary part and y is coefficient of the imaginary part. The set of complex numbers includes the set of real numbers, and it is denoted by C.

(2) Properties of Complex Numbers

- If $z = x + iy$, then the real part of z is denoted by Re (z) and the imaginary part by Im (z).
- A complex number is said to be purely imaginary if $Re(z) = 0$.
- A complex number is said to be purely real if $\text{Im}(z) = 0$.
- The complex number $0 = 0 + i0$ is both purely real and purely imaginary.
- Two complex numbers are said to be equal if and only if their real parts and imaginary parts are separately equal, i.e.,

  $x_1 + iy_1 = x_2 + iy_2$ implies

  $x_1 = x_2$ and $y_1 = y_2$.
- However, there is no order relation between complex and the expressions of the type

  $x_1 + iy_1 < (or >) x_2 + iy_2$ are meaningless.

(3) Properties of Argument

- $\arg(z_1 z_2) = \varphi_1 + \varphi_2 = \arg(z_1) + \arg(z_2)$
- $\arg\left(\dfrac{z_1}{z_2}\right) = \varphi_1 - \varphi_2 = \arg(z_1) - \arg(z_2)$
- $\arg(z^n) = n\arg(z), n \in I$

  In the above result $\varphi_1 + \varphi_2$ or $\varphi_1\varphi_2$ are not necessarily the principle values of the argument of corresponding complex numbers.
- $\arg(z) = 0, \pi, \Rightarrow z$ is a purely real number

  $\Rightarrow z = \overline{z}$
- $\arg(z) = \dfrac{\pi}{2}, \dfrac{-\pi}{2} \Rightarrow z$ is a purely imaginary number

  $\Rightarrow z = -\overline{z}$

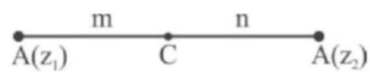
(4) De Moivre's Theorem

Theorem (De Moivre's Theorem).

$(\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi$

(5) Section Formula

If points $A(z_1)$ and $B(z_2)$ represent the complex numbers z_1 and z_2 respectively in the Argand plane, then:



$C \equiv \left(\dfrac{mz_2 + nz_1}{m + n}\right)$ is the point dividing AB in the ratio $m : n$.

(6) Important Properties

(I) $|z_1 \cdot z_2 \cdot z_3 \ldots\ldots z_n| = |z_1||z_2||z_3| \ldots\ldots |z_n|$.

$\arg(z_1 \cdot z_2 \cdot z_3 \ldots\ldots z_n) = \arg(z_1) + \arg(z_2) + \ldots\ldots + \arg(z_n)$

(II) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

(III) **(i)** $\overline{z_1 + z_2 + z_3 + \ldots\ldots + z_n} = \overline{z_1} + \overline{z_2} + \overline{z_3} + \ldots\ldots + \overline{z_n}$

(ii) $\overline{z_1 \cdot z_2 \cdot z_3 \ldots\ldots z_n} = \overline{z_1} \cdot \overline{z_2} \cdot \overline{z_3} \ldots\ldots \overline{z_n}$

(iii) $\left(\overline{\dfrac{z_1}{z_2}}\right) = \dfrac{\overline{z_1}}{\overline{z_2}}$

(iv) $\left(\overline{z^n}\right) = \left(\overline{z}\right)^n$

(IV) $z + \overline{z} = 2\text{Re}(z) \quad \Rightarrow \quad \overline{z} = -z$

if z is purely imaginary $(\because Re(z) = 0)$

$z - \overline{z} = 2i\,\text{Im}(z) \quad \Rightarrow \quad \overline{z} = z$

if z is purely real $(\because Im(z) = 0)$

(V) $z\overline{z} = |z|^2 \quad \Rightarrow \quad \overline{z} = \dfrac{1}{z}\, if\, |z| = 1$

(VI) $|-z| = |\overline{z}| = |z|$ and $\arg(\overline{z}) = -\arg(z)$

(VII) $\left|z^n\right| = |z|^n$

(VIII) $(z - z_0)$ is a factor of $f(z)$ *if and only if*

$f(z_0) = 0$

- $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2\,\mathrm{Re}(z_1\bar{z}_2) = 2\,\mathrm{Re}(\bar{z}_1 z_2)$

  $\Rightarrow \quad z_1\bar{z}_2 + \bar{z}_1 z_2$ is purely real

- $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + \bar{z}_1 z_2)$

  $= |z_1|^2 + |z_2|^2 \pm 2\,\mathrm{Re}(z_1\bar{z}_2) = |z_1|^2 + |z_2|^2 \pm 2\,\mathrm{Re}(\bar{z}_1 z_2)$

- $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

- $-|z| \le \mathrm{Re}(z) \le |z|, -|z| \le \mathrm{Im}(z) \le |z|$

- Triangle Inequality :

  (i) $|z_1 + z_2| \le |z_1| + |z_2|$

  (ii) $|z_1 - z_2| \ge \big||z_1| - |z_2|\big|$

- $\dfrac{e^{i\theta} - 1}{e^{i\theta} + 1} = i\tan\dfrac{\theta}{2}$

(7) The n^{th} Root of Unity

(I) Properties of n roots of Unity Properties:

- Sum of n roots unity is zero

$$1 + \omega + \omega^2 + \ldots + \omega^{n-1} = \frac{1 - \omega^n}{1 - \omega} = 0$$

$$\Rightarrow \sum_{k=0}^{n-1} \cos\frac{2k\pi}{n} = 0 \text{ and } \sum_{k=0}^{n-1} \sin\frac{2k\pi}{n} = 0$$

Thus the sum of the roots of unity is zero.

- Sum of p^{th} power of n roots of unity is zero, if p is not a multiple of n

$$1 + \omega^p + (\omega^2)^p + \ldots + (\omega^{n-1})^p = \frac{1 - (\omega^n)^p}{1 - \omega^p}$$

$$= \frac{1 - \left(e^{i\frac{2p\pi}{n}}\right)^n}{1 - \omega^p}$$

$$= \frac{1 - (e^{i-2p\pi})}{1 - \omega^p}$$

- Sum of p^{th} power of n roots of unity is n, if p is a multiple of n

Let $p = \lambda n$, thus

$$\omega^p = e^{i\frac{2\pi p}{n}} = e^{i2\pi\lambda} = (\cos 2\pi\lambda + i\sin 2\pi\lambda) = 1$$

$$1 + \omega^p + (\omega^2)^p + \ldots + (\omega^{n-1})^p = \frac{1 - (\omega^n)^p}{1 - \omega^p}$$

$$= 1 + 1 + 1 + \ldots (n\ times) = n$$

- Product of the roots

1. $\omega^p \cdot (\omega^2)^p \ldots (\omega^{n-1})^p = \omega^{\frac{n(n-1)}{2}}$

$$= \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^{\frac{n(n-1)}{2}}$$

$$= \cos(n-1)\pi + i\sin(n-1)\pi$$

If n is even, $\omega^{\frac{n(n-1)}{2}} = -1$ and in case n is odd $\omega^{\frac{n(n-1)}{2}} = 1$

- The points represented by the $n\ n^{th}$ roots of unity are located at the vertices of a regular polygon of n sides inscribed in a unit circle having centre at the origin, one vertex being on the positive real axis (Geometrically represented as shown)

(II) Cube roots of unity:

For $n = 3$, we get the cube roots of unity and they are

$$1, \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} \text{ and}$$

$$\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} \text{ i.e., } 1, \frac{-1 + i\sqrt{3}}{2} \text{ and}$$

$\dfrac{-1 - i\sqrt{3}}{2}$. They are generally denoted by $1, \omega$ and ω^2 and are geometrically represented by the vertices of an equilateral triangle whose circumcentre is the origin and circumradius is unity.

(8) Properties of Cube Roots of Unity

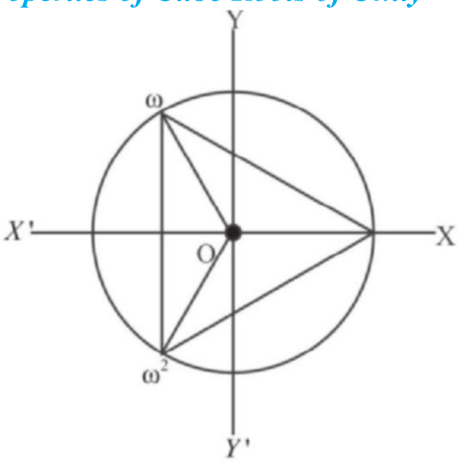


Fig-10

- $\omega^3 = 1$
- $1+\omega+\omega^2 = 0$
- $1+\omega^n+\omega^{2n} = 3,$ where n is multiple of 3.
- $1+\omega^n+\omega^{2n} = 3$, (n is an integer, not a multiple of 3).
- $\omega = \frac{1}{\omega^2}$ and $\omega^2 = \frac{1}{\omega}$.
- $\omega = (\omega^2)^2$
- $\overline{\omega} = \omega^2$ and $\omega^2 = \overline{\omega}$

PERMUTATIONS & COMBINATIONS

(1) Permutations

A permutation is an ordering of a list of objects.
Theorem: The number of permutations of n distinct is $n!$, the factorial of n.

(I) Permutations of a subset of Distinct Objects:
If we have n objects and want to arrange k of them in a row, there are $\frac{n!}{(n-k)!}$ ways to do this. This is also known as a k-permutation of n, and is denoted by ${}^{n}p_k$.

(II) Permutations with Restrictions

- The number of ways to select and arrange (permutate) r objects from n different objects such that arrangement should always include p particular objects is ${}^{n-p}C_{r-p}\underline{\lfloor r}$.

  The number of ways to select and arrange r objects from n different objects such that p particular objects are always excluded in the selection is ${}^{n-p}C_r\underline{\lfloor r}$.

  The number of ways to arrange n different objects such that p particular objects are always separated is ${}^{n-p+1}C_p\underline{\lfloor (n-p)}\cdot\underline{\lfloor p}$

(2) Restrictions to Adjacent objects

The number of ways to arrange n different objects such that p particular objects remain together in the arrangement is

$\underline{\lfloor n-p+1}\,\underline{\lfloor p}$

(3) Combinations

A combination is a way of choosing elements from a set in which order does not matter.
In general, the number of ways to pick r unordered elements from an n element set is

$\frac{n!}{r!(n-r)!}$. This is a [bionomial coefficient],

denoted $\binom{n}{r}$.

Properties of ${}^{n}C_r$

- ${}^{n}C_0 = {}^{n}C_n = 1$
- ${}^{n}C_r = {}^{n}C_{n-r}$
- If ${}^{n}C_r = {}^{n}C_k$, then $r = k$ or $n - r = k$
- ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$ **(v)** $r.{}^{n}C_r = n^{n-1}C_{r-1}$

- $\frac{1}{r+1}{}^nC_r = \frac{1}{n+1}{}^{n+1}C_{r+1}$
- $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$
- (a) If n is even, ${}^nC_r$ is greatest for $r = n/2$.

  (b) If n is odd, ${}^nC_r$ is greatest for $r = \frac{n-1}{2}, \frac{n+1}{2}$

(4) Identical Objects into Distinct Bins

Theorem: Suppose there are n identical objects to be distributed among r distinct bins. This can be done in precisely

$\binom{n+r-1}{r-1}$ ways.

Theorem: Suppose there are n distinct objects, of which k are to be distributed among r distinct bins. This can be done in precisely $\binom{n}{k}r^k$ ways.

(5) Always including particular objects in the selection

The number of ways to select r objects from n different objects where p particular objects should always be included in the selection is ${}^{n-p}C_{r-p}$.

(6) Always excluding p particular objects in the selection

The number of ways to select r objects from n different objects where p particular objects should never be included in the selection is ${}^{n-p}C_r$.

(7) Problems based on atleast or atmost constraint

There are problems in which constraints are to select minimum (at least) or maximum (at most) objects in the selection.

(8) Division of Identical Objects into Groups

Here, we will define the formula to find number of always to divide identical objects into groups.

- The number of ways to divide n identical objects into r groups (different) such that each group gets 0or more objects (empty groups are allowed) ${}^{n+r-1}C_{r-1}$.
- The number of ways to divide n identical objects into r groups (different) such that each group receives at least one object (empty groups are notallowed) is ${}^{n-1}C_{r-1}$.
- The number of ways to divide n identical objects in r groups (different) such that each group gets minimum m objects and maximum k objects = Coefficient of x^n in $(x^m + x^{m+1} + ... + x^k)^r$.

(9) Unequal division and distribution of non-identical objects

- Number of ways in which $(m+n+p)$ different objects can be divided into 3 unequal groups (groups contain unequal number of objects) containing m, n, p objects ${}^{m+n+p}C_m \, {}^{n+p}C_n \, {}^pC_p$

  $$= \frac{(m+n+p)}{m!n!p!}.$$
- Number of ways in which $(m+n+p)$ different objects can be divided and distributed (entries are distributed among groups) into 3 unequal groups (groups contain unequal number of objects) containing m, n, p objects = No. of ways to divide $(m+n+p)$ objects in 3 groups × No. of ways to distribute 'division- ways'among groups = No. of ways to divide $(m+n+p)$ objects in 3 groups × (Number of groups)

  $$!= \frac{(m+n+p)}{m!n!p!} \times 3!.$$

(10) Equal division and distribution of non-identical objects

Here, we will see formulae to divide and distribute non-identical objects equally in groups i.e., each group gets equal numbers of objects.

(11) Equal as well as Unequal Division and Distribution of non-identical objects

- Number of ways to divide $(m+2n+3p)$ objects in 6 groups containing m, n, n, p, p, p objects is:

  $$\frac{\lfloor m+2n+3p}{\lfloor m(\lfloor n)^2(\lfloor p)^3} \times \frac{1}{\lfloor 2} \times \frac{1}{\lfloor 3}$$

(12) Selection of one or more objects from n different objects

The number of ways to select one or more objects from n different objects or we can say, selection of at least one object from n different objects $2^n - 1$.

(13) Selection of one or more objects from n identical objects

The number of ways to select one or more objects (or at least one object) from n identical objects is = n

(14) Selection of one or more objects from objects which are not all different from each other

The number of ways to select one or more objects from $(p+q+r.......+n)$ objects where p objects are alike of one kind, q are alike of second kind, r are alike of third kind, ... and remaining n are different from each other is

$$\left[2^n(p+1)(q+1)(r+1)...\right]-1.$$

(15) Dearrangement Theorem

If n distinct objects are to be arranged in a row such that no object occupies its original place, then to find number of ways to arrange them, we use dearrangement theorem i.e.,

Number of ways to dearrange

$$= \lfloor n\left[1-\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+......(-1)^n\frac{1}{\lfloor n}\right]$$

(16) Rank of a word in the dictionary

In this problem type, dictionary of words is formed by using the letters of the given word.

The dictionary format means words are arranged in the alphabetical order. You will be supposed to find the rank (position)of the given word or some other word in the dictionary.

(17) Points of Intersection between geometrical figures

We can use ${}^nC_r$ (number of ways to select r objects from n different objects) to find points of intersection between geometrical figures.

- Number of points of intersection between n non-concurrent and non-parallel lines is ${}^nC_1$.
- Number of lines that can be drawn using n points such that no three of them are collinear is ${}^nC_2$.
- Number of triangles that can be formed using n points such that no three of them are collinear is ${}^nC_3$.
- Number of diagonals that can be drawn in a n sided polygon is $\frac{n(n-3)}{2}$.

Exercise

Matrices

1. If $A=\begin{bmatrix}1&0&0\\0&1&0\\a&b&-1\end{bmatrix}$ and I is the unit matrix of order 3, then $A^2+2A^4+4A^6$ is equal to

(a) $6I$ (b) $8I$ (c) $7A^7$ (d) $7A^8$

2. Let $A=\begin{bmatrix}0&\alpha\\0&0\end{bmatrix}$ and $(A+I)^{50}-50A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$.

Then the value of $a+b+c+d$ is

(a) 1 (b) 2 (c) 4 (d) None of these

3. If A and B are square matrix of the same order, then $(A+B)^2 = A^2+2AB+B^2$ implies

(a) $AB=BA$ (b) $AB+BA$

(c) $AB=0$ (d) none of these

4. If $A=\begin{bmatrix}\cos\alpha&\sin\alpha\\-\sin\alpha&\cos\alpha\end{bmatrix}$ then AA^T =

(a) $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ (b) $\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$

(c) $\begin{bmatrix}5&0&-2\\0&6&0\\-2&6&2\end{bmatrix}$ (d) $\begin{bmatrix}0&0&0\end{bmatrix}$

5. If three matrices A, B and C satisfy associative law, then

(a) $A(BC)=(AB)C$ (b) $A(BC)=(AC)B$
(c) $A+BC=AB+C$ (d) $A-BC=AC-B$

6. The matrix A has 'x' rows and $(x+5)$ columns. The matrix B has 'y' rows and $(11-y)$ columns. Both AB and BA exist. The values x and y are
(a) 8,3 (b) 13,4 (c) 3,8 (d) 8,8

7. For a given $A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ which of the following statements holds good ?
(a) $A=A^{-1}, \forall\theta\in R$
(b) A is symmetric, for $\theta=(2n+1)\frac{\pi}{2}, n\in z$
(c) A is an orthogonal matrix, for $\theta\in R$
(d) A is Skew Symmetric, for $\theta=n\pi, n\in z$

Determinants

8. The value of a 3×3 determinant is 3, value of determinant formed by its co-factor is
(a) 3 (b) 6 (c) 9 (d) 27

9. If a, b, c are in A.P, then

$$\begin{vmatrix}2y+4 & 5y+7 & 8y+a\\ 3y+5 & 6y+8 & 9y+b\\ 4y+6 & 7y+9 & 10y+c\end{vmatrix}$$ is equal to

(a) $a+b+c$ (b) 0 (c) 1 (d) $2y^3$

10. If $\begin{vmatrix}p & q-y & r-z\\ p-x & q & r-z\\ p-x & q-y & r\end{vmatrix}=0$ then the value of $\frac{p}{x}+\frac{q}{y}+\frac{r}{z}$ is
(a) 0 (b) 1 (c) 2 (d) 3

11. If A and B are square matrices of order 3, such that $|A|=-1, |B|=3$ then the determinant of $3AB$ is equal to
(a) 81 (b) -9 (c) -81 (d) -27

12. If $f(x), g(x)$ and $h(x)$ are three polynomials of degree '2' and

$$\phi(x)=\begin{vmatrix}f(x) & g(x) & h(x)\\ f'(x) & g'(x) & g'(x)\\ f''(x) & g''(x) & h''(x)\end{vmatrix}$$, then $\phi'(x)$ is

(a) A one degree polynomial
(b) A three degree polynomial
(c) A two degree polynomial
(d) A constant

13. If

$$\begin{vmatrix}(1+x) & (1+x)^2 & (1+x)^3\\ (1+x)^4 & (1+x)^5 & (1+x)^6\\ (1+x)^7 & (1+x)^8 & (1+x)^9\end{vmatrix}=a_0+a_1x+a_2x^2+.....,$$

then, a_1 is equal to
(a) 1 (b) 2 (c) 0 (d) 3

14. If the three linear equations
$x+4ay+az=0$
$x+3by+bz=0$
and $x+2cy+cz=0$ has a non-trivial solution, where $a\neq0, b\neq0$ and $c\neq0$, then $ab+bc$ is equal to
(a) $2ac$ (b) $-ac$ (c) ac (d) $-2ac$

Principles of Mathematical Induction

15. $10^n+3.4^{n+2}+k$ is divisible by 9 for $n\in N$. Then, the least positive integral value of k is
(a) 1 (b) 3 (c) 5 (d) 7

16. $x(x^{n-1}-n\,a^{n-1})+a^n(n-1)$ is divisible by $(x-a)^2$ for
(a) $n>1$ (b) $n>2$
(c) $\forall n\in N$ (d) none of these

17. If $n\in N$,then $7^{2n}+3^{3n-3}.3^{n-1}$ is always divisible by
(a) 25 (b) 35 (c) 45 (d) None of these

Theory of Equations

18. If $\sin^2\alpha\cdot\cos^2\alpha=\sin^2\beta$ then the roots of the equation $x^2+2x\cot\beta+1=0$ are always
(a) Equal (b) Imaginary
(c) Real and distinct (d) Greater than 1

19. If $a,b,c\in R$ and $(a+b+c)\,c<0$, then the quadratic equation $f(x)=ax^2+bx+c=0$ has:
(a) A negative root (b) Two real root
(c) Two imaginary root (d) None of these

20. The least Integral value of 'k' for which $(k-2)x^2+8x+k+4>0$ for all $x\in R$, is :

(a) 5 (b) 2 (c) 3 (d) None of these

21. If are roots $a=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$,

$\alpha=a+a^2+a^4$ and $\beta=a^3+a^5+a^6$

then α,β are roots of the equation

(a) $x^2+x+1=0$ (b) $x^2+x+2=0$

(c) $x^2+2x+2=0$ (d) $x^2+2x+3=0$

22. The condition that the roots of $\frac{x-a}{ax+1}-\frac{x+b}{bx+1}=0$ are reciprocal to each other is

(a) $a=0$ (b) $a+b=0$

(c) $a-b=0$ (d) $b=0$

Complex Numbers

23. The real part of $e^{e^{i\theta}}$ is

(a) $e^{\cos\theta}[\cos(\sin\theta)]$ (b) $e^{\cos\theta}[\mathrm{cis}(\cos\theta)]$

(c) $e^{\sin\theta}[\sin(\sin\theta)]$ (d) $e^{\sin\theta}[\sin(\cos\theta)]$

24. The imaginary part of

$(z-1)(\cos\alpha-i\sin\alpha)+(z-1)^{-1}\times(\cos\alpha+i\sin\alpha)$

is zero, if :

(a) $|z-1|=2$ (b) $\arg(z-1)=2\alpha$

(c) $\arg(z-1)=\alpha$ (d) None of these

25. The argument of the complex number $\left(\frac{i}{2}-\frac{2}{i}\right)$ is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{12}$ (d) $\frac{\pi}{2}$

26. The modulus and amplitude of $(1+i\sqrt{3})^8$ are respectively

(a) 256 and $\pi/3$ (b) 256 and $2\pi/3$

(c) 2 and $2\pi/3$ (d) 256 and $8\pi/3$ s

27. The complex number $Z=\begin{vmatrix}2 & 3+i & -3\\ 3-i & 0 & -1+i\\ -3 & -1-i & 4\end{vmatrix}$ is

(a) $3-4i$ (b) $5+4i$

(c) $-5i$ (d) A real number

28. If $2z=7+i\sqrt{3}$, then the value of $\left(z^2-7z+1\right)^2-2\left(z^2-7z\right)$ is equal to

(a) -118 (b) 145 (c) 170 (d) 197

29. If z is a complex number such that $z+\sqrt{2}|z+1|+i=0$ then $|z|^2$ is equal to

(a) 5 (b) 3 (c) 1 (d) None of these

30. Equation of the circle described on the ends of the diameter z_1 and z_2 is

(a) $\mathrm{Re}\{(\bar{z}-\bar{z}_1)(z-z_2)\}=0$

(b) $\mathrm{Im}\{(z-\bar{z}_1)(z-\bar{z}_2)\}=0$

(c) $\mathrm{Re}\{(z-\bar{z}_1)(z-\bar{z}_2)\}=0$

(d) None of these

Permutations & Combinations

31. All the words that can be formed using alphabets A, H, L, U, R are written as in a dictionary (no alphabet is repeated). Then, the rank of the word 'RAHUL' is

(a) 70 (b) 71 (c) 74 (d) 73

32. A person always prefers to eat paratha and vegetable dish in his meal. How many ways can he make his platter in a marriage party if there are three types of parathas, four types of vegetable dish, three types of salads, and two types of sauces ?

(a) 3360 (b) 4096 (c) 3000 (d) None of these

33. The number of differnt seven digit numbers that can be written using only three digits 1, 2 and 3 with the condition that the digit 2 occurs twice in each number is

(a) $7P_2\cdot 2^5$ (b) $7C_2\cdot 2^5$ (c) $7C_2\cdot 5^2$ (d) $7C_2\cdot 2^7$

34. The number of different pairs of words that can be made with the letters of the word STATICS is

(a) 828 (b) 1260 (c) 396 (d) None of these

35. The number of ways in which 12 books can be put in three shelves with four on each shelf is

(a) $\frac{12!}{(4!)^3}$ (b) $\frac{12!}{(3!)(4!)^3}$ (c) $\frac{12!}{(3!)^3\,4!}$ (d) None of these

ANSWER KEY

1. d	2. b	3. a	4. a	5. a
6. c	7. c	8. c	9. b	10. c
11. c	12. d	13. c	14. a	15. c
16. c	17. a	18. c	19. b	20. a
21. b	22. d	23. b	24. c	25. d
26. b	27. d	28. c	29. a	30. a
31. c	32. a	33. b	34. b	35. a

CLASS XII MATHEMATICS KVPY-7
PREVIOUS YEAR QUESTIONS

Functions

1. Let R be a relation on the set of all natural numbers given by $aRb \Rightarrow a$ divides b^2. Which of the following properties does R satisfy? **[2017]**

I.Reflexivity
II. Symmetry
III.Transitivity

(a) I only (b) III only
(c) I and III only (d) I and II only

2. Let X be a set of 5elements. The number d of ordered pairs (A,B) of subsets of X such that $A \neq \phi$, $B \neq \phi$, $A \cap B = \phi$ satisfies **[2014]**

(a) $50 \le d \le 100$ (b) $101 \le d \le 150$
(c) $151 \le d \le 200$ (d) $201 \le d$

3. For an integer n let $S_n = \{n+1, n+2, \ldots, n+18\}$. Which of the following is true for all $n \ge 10$? **[2013]**

(a) S_n has a multiple of 19
(b) S_n has a prime
(c) S_n has at least four multiples of 5
(d) S_n has at most six primes

4. Let $S = \{1,2,3,\ldots n\}$ and $A = \{(a,b) | 1 \le a, b \le n\} = S \times S$. A subset B of A is said to be a good subset if $(x,x) \in B$ for every $x \in S$. Then the number of good subsets of A is- **[2012]**

(a) 1 (b) 2^n (c) $2^{n(n-1)}$ (d) 2^{n^2}

5. Let $f(x) = \cos 5x + A\cos 4x + B\cos 3x + C\cos 2x + D\cos x + E$, and

$$T = f(0) - f\left(\frac{\pi}{5}\right) + f\left(\frac{2\pi}{5}\right) - f\left(\frac{3\pi}{5}\right) + \ldots + f\left(\frac{8\pi}{5}\right) - f\left(\frac{9\pi}{5}\right)$$

.Then T **[2011]**

(a) Depends on A, B, C, D, E
(b) Depends on A, C, E but independent of B and D
(c) Depends on B, D but independent of A, C, E.
(d) Is independent of A, B, C, D, E

6. Let X be a nonempty set and let $P(X)$ denote the collection of all subsets of X. Define $f: X \times P(X) \to R$ by

$$f(x,A) = \begin{cases} 1, & if \quad x \in A \\ 0, & if \quad x \notin A \end{cases}$$

Then $f(x, A \cup B)$ equals **[2011]**

(a) $f(x,A) + f(x,B)$
(b) $f(x,A) + f(x,B) - 1$
(c) $f(x,A) + f(x,B) - f(x,A)f(x,B)$
(d) $f(x,A) + |f(x,A) - f(x,B)|$

7. Let $[x]$ denote the largest integer not exceeding x and $\{x\} = x - [x]$ .Then $\int_0^{2012} \frac{e^{\cos(\pi\{x\})}}{e^{\cos(\pi\{x\})} + e^{-\cos(\pi\{x\})}} dx$ is equal to **[2011]**

(a) 0 (b) 1006 (c) 2012 (d) 2012π

8. Which of the following intervals is a possible domain of the function $f(x) = \log_{\{x\}}[x] + \log_{[x]}\{x\}$,where $[x]$ is the greatest integer not exceeding x and $\{x\} = x - [x]$? **[2011]**

(a) (0, 1) (b) (1, 2) (c) (2, 3) (d) (3, 5)

Matrices and Determinants

1. For how many different values a does the following system have at least two distinct solutions?

$$ax + y = 0$$

$x+(a+10)y=0$ **[2017]**

(a) 0 (b) 1
(c) 2 (d) Infinitely many

2. The remainder when the determinant

$$\begin{vmatrix} 2014^{2014} & 2015^{2014} & 2016^{2016} \\ 2017^{2017} & 2018^{2018} & 2019^{2019} \\ 2020^{2020} & 2021^{2021} & 2022^{2022} \end{vmatrix}$$ is divided by 5 is

[2015]

(a) 1 (b) 2 (c) 3 (d) 4

3. Let P be an $m \times m$ matrix such that $P^2 = P$. Then $(1+P)^n$ equals **[2011]**

(a) $I+P$ (b) $I+nP$
(c) $I+2^n P$ (d) $I+(2^n-1)P$

4. Let A and B be any two $n \times n$ matrices such that the following conditions hold : $AB = BA$ and there exist positive integers k and ℓ such that $A^k = I$ (the identity matrix) and $B^\ell = 0$ (the zero matrix). Then **[2011]**

(a) $A+B=I$
(b) det $(AB)=0$
(c) det $(A+B) \neq 0$
(d) $(A+B)^m = 0$ for some integer m

5. Let A denote the matrix $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, where $i^2=-1$, and let I denote the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $I+A+A^2+\ldots A^{2010}$ is **[2010]**

(a) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ (d) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Binomial Theorem

1. The fractional part of a real number x is $x-[x]$, where $[x]$ is the greatest integer less than or equal to x. Let F_1 and F_2 be the fractional parts of $(44-\sqrt{2017})^{2017}$ and $(44+\sqrt{2017})^{2017}$ respectively. Then F_1+F_2 lies between the numbers **[2017]**

(a) 0 and 0.45 (b) 0.45 and 0.9
(c) 0.9 and1.35 (d) 1.35 and 1.8

2. Let $x=(\sqrt{50}+7)^{1/3}-(\sqrt{50}-7)^{1/3}$. Then **[2015]**

(a) $x=2$
(b) $x=3$
(c) x is a rational number ,but not an integer
(d) x is an irrational number

3. Let $(1+x+x^2)^{2014} = a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_{4028}x^{4028}$, and let

$A = a_0 - a_3 + a_6 - \cdots + a_{4026}$,

$B = a_1 - a_4 + a_7 - \cdots - a_{4027}$,

$C = a_2 - a_5 + a_8 - \cdots + a_{4028}$,

Then **[2015]**

(a) $|A|=|B|>|C|$ (b) $|A|=|B|<|C|$
(c) $|A|=|C|>|B|$ (d) $|A|=|C|<|B|$

4. The coefficient of x^{2012} in $\dfrac{1+x}{(1+x^2)(1-x)}$ is **[2012]**

(a) 0 (b) 2011 (c) 2012 (d) 1

5. Arrange the expansion of $\left(x^{1/2}+\dfrac{1}{2x^{1/4}}\right)^n$ in decreasing powers of x. Suppose the coefficient of the first three terms form an arithmetic progression. Then the number of terms in the expansion having integer powers of x is- **[2010]**

(a) 1 (b) 2
(c) 3 (d) More than 3

Inequalities

1. Suppose p, q, r are real numbers such that $q=p(4-p), r=q(4-q), p=r(4-r)$. The maximum possible value of $p+q+r$ is **[2017]**

(a) 0 (b) 3 (c) 9 (d) 27

2. Let $a,b,c,d,e,$ be real numbers such that a $a+b<c+d, b+c<d+e, c+d<e+a, d+e<a+b.$ Then **[2017]**

(a) The largest is a and the smallest is b

(b) The largest is a and the smallest is c
(c) The largest is c and the smallest is e
(d) The largest is c and the smallest is b

3. Let f be a continuous function defined on [0, 1] such that $\int_0^1 f^2(x)\,dx = \left(\int_0^1 f(x)\,dx\right)^2$. Then the range of **[2016]**

(a) Has exactly two points
(b) Is the interval [0,1]
(c) Has more than two points
(d) Is a singleton

4. For a real number r we denote by $[r]$ the largest integer less than or equal to r. If x, y are real numbers with $x, y \geq 1$ then which of the following statements is always true ? **[2014]**

(a) $[x+y]\cdot[x]+[y]$ (b) $[xy] \leq [x][y]$

(c) $\left[2^x\right] \leq 2^{[x]}$ (d) $\left[\frac{x}{y}\right] \leq \frac{[x]}{[y]}$

5. For a real number r let $[r]$ denote the largest integer less than or equal to r. Let $a > 1$ be a real number which is not an integer and let k be the smallest positive integer such that $\left[a^k\right] > [a]^k$. Then which of the following statements is always true ? **[2014]**

(a) $k \leq 2([a]+1)^2$ (b) $k \leq ([a]+1)^4$

(c) $k \leq 2^{|a|+1}$ (d) $k \leq \frac{1}{a-[a]}+1$

6. In a triangle ABC, let G denote its centroid and let M, N be points in the interiors of the segments AB, AC, respectively, such that M, G, N are collinear. If r denotes the ratio of the area of triangle AMN to the area of ABC then **[2013]**

(a) $r = \frac{1}{2}$ (b) $r > \frac{1}{2}$

(c) $\frac{4}{9} \leq r < \frac{1}{2}$ (d) $\frac{4}{9} < r$

7. The maximum value M of $3^x + 5^x - 9^x + 15^x - 25^x$, as x varies over reals, satisfies- **[2012]**

(a) $3 < M < 5$ (b) $0 < M < 2$
(c) $9 < M < 25$ (d) $5 < M < 9$

8. The minimum value of n for which

$$\frac{2^2+4^2+6^2+\ldots+(2n)^2}{1^2+3^2+5^2+\ldots+(2n-1)^2} < 1.01$$

(a) Is 101 (b) Is 121
(c) Is 151 (d) Does not exist

Complex Numbers

1. For any real number r, let $A_r = \{e^{i\pi rn} : n \text{ is a natural number}\}$ be a set of complex numbers. Then- **[2016]**

(a) $A_1, A_{\frac{1}{\pi}}, A_{0.3}$ are all infinite sets

(b) A_1 is a finite set and $A_{\frac{1}{\pi}}, A_{0.3}$ are infinite sets

(c) $A_1, A_{\frac{1}{\pi}}, A_{0.3}$ are all finite sets

(d) $A_1, A_{0.3}$ are finite sets and $A_{\frac{1}{\pi}}$, is an infinite sets

2. Let a be a fixed non-zero complex number with $|a| < 1$ and $w = \left(\frac{z-a}{1-\bar{a}z}\right)$. Where z is a complex number. **[2016]**

(a) There exists a complex number z with $|z| < 1$ such that $|w| > 1$

(b) $|w| > 1$ for all z such that $|z| < 1$

(c) $|w| < 1$ for all z such that $|z| < 1$

(d) There exists z such with $|z| < 1$ and $|w| = 1$

3. If z is a complex number satisfying $\left|z^3 + z^{-3}\right| \leq 2$, then the maximum possible value of $\left|z + z^{-1}\right|$ is **[2015]**

(a) 2 (b) $\sqrt[3]{2}$ (c) $2\sqrt{2}$ (d) 1

4. If n is a positive integer and $\omega \neq 1$ is a cube root of unity, the number of possible values of $e^{\left|\sum_{k=0}^{n}\binom{n}{k}\omega^k\right|}$

[2014]

(a) 2 (b) 3 (c) 4 (d) 6

5. Let ω be a cube root of unity not equal to 1. Then the maximum possible value of $|a+b\omega+c\omega^2|$ where $a,b,c\in\{+1,-1\}$ is [2013]

(a) 0 (b) 2 (c) $\sqrt{3}$ (d) $1+\sqrt{3}$

6. Suppose n is a natural number such that $|i+2i^2+3i^3+...+ni^n|=18\sqrt{2}$, where i is the square root of -1. Then n is- [2011]

(a) 9 (b) 18 (c) 36 (d) 72

7. Let I,ω and ω^2 be the cube roots of unity. The least possible degree of a polynomial, with real coefficients, having $2\omega^2, 3+4\omega, 3+4\omega^2$ and $5-\omega-\omega^2$ as roots is- [2010]

(a) 4 (b) 5 (c) 6 (d) 8

Three Dimensional Geometry

1. The shortest distance from the origin to a variable point on the sphere $(x-2)2+(y-3)^2+(z-6)^2=1$ is [2015]

(a) 5 (b) 6 (c) 7 (d) 8

Coordinate System

1. Let BC be a fixed line segment in the plane. The locus of a point A such that the triangle ABC is isosceles, is (with finitely many possible exceptional points) [2017]

(a) A line
(b) A circle
(c) The union of a circle and a line
(d) The union of two circles and a line

2. Let $A=(a_1,a_2)$ and $B=(b_1,b_2)$ be two points in the plane with integer coordinates. Which one of the following is not a possible value of the distance between A and B ? [2017]

(a) $\sqrt{65}$ (b) $\sqrt{74}$ (c) $\sqrt{83}$ (d) $\sqrt{97}$

3. Let $O=(0,0)$; let A and B be points respectively on x-axis and y-axis such that $\angle OBA=60°$. Let D be a point in the first quadrant such that OAD is an equilateral triangle. Then the slope of DB is [2016]

(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{\sqrt{3}}$

4. The sides of a right-angled triangle are integers. The length of one of the sides is 12. The largest possible radius of the incircle of such a triangle is [2015]

(a) 2 (b) 3 (c) 4 (d) 5

5. Let $b,d>0$. The locus of all points $P(r,\theta)$ for which the line OP (where O is the origin) cuts the linear $r\sin\theta=b$ in Q such that $PQ=d$ is [2014]

(a) $(r-d)\sin\theta=b$ (b) $(r\pm d)\sin\theta=b$

(c) $(r-d)\cos\theta=b$ (d) $(r\pm d)\cos\theta=b$

6. Let ABC be a triangle such that $AB=BC$. Let F be the midpoint of AB and X be a point on BC such that FX is perpendicular to AB. If $BX=3XC$ then the ratio BC/AC equals [2014]

(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) $\sqrt{\frac{3}{2}}$ (d) 1

7. If a, b are positive real numbers such that the lines $ax+9y=5$ and $4x+by=3$ are parallel, then the least possible value of $a+b$ is [2013]

(a) 13 (b) 12 (c) 8 (d) 6

8. Consider a triangle ABC in the xy-plane with vertices $A=(0,0), B=(1,1)$ and $C=(9,1)$ If the line $x=a$ divides the triangle into two parts of equal area, then a equals [2013]

(a) 3 (b) 3.5 (c) 4 (d) 4.5

9. Let $A=(4,0), B=(0,12)$ be two points in the plane. The locus of a point C such that the area of triangle ABC is 18 sq. units is [2011]

(a) $(y+3x+12)^2=81$ (b) $(y+3x+81)^2=12$

(c) $(y+3x-12)^2=81$ (d) $(y+3x-81)^2=12$

10. In triangle ABC, we are given that $3\sin A+4\cos B=6$ and $4\sin B+3\cos A=1$. Then the measure of the angle C is [2011]

(a) 30° (b) 150° (c) 60° (d) 75°

11. Let ABC be an equilateral triangle, let $KLMN$ be a rectangle with K, L on BC, M on AC and N on AB. Suppose $AN/NB=2$ and the area of triangle BKN is 6. The area of the triangle ABC is [2010]

(a) 54
(b) 108
(c) 48
(d) Not determinable with the above data

ANSWER KEY

Functions

1. a	2. c	3. d	4. c	5. c
6. c	7. b	8. c		

Matrices and Determinants

1. c	2. d	3. d	4. d	5. c

Binomial Theorem

1. c	2. a	3. d	4. d	5. c

Inequalities

1. c	2. a	3. d	4. d	5. b
6. c	7. b	8. c		

Complex Numbers

1. d	2. c	3. a	4. c	5. b
6. c	7. b			

Three Dimensional Geometry

1. b

Coordinate System

1. d	2. c	3. d	4. d	5. b
6. c	7. b	8. a	9. c	10. a
11. b				

HINTS & SOLUTIONS

Functions

1.Sol: (I) This relation is reflexive relation because every natural number divides square of it self a R a $\Leftrightarrow$ a divides a^2

(II) Not symmetric eg. 5R 10 $\Rightarrow$ 5 Divide 100 But 10 R 5 $\Rightarrow$ 10Divide 25

(III) Not transitivity for example if 8 R 4&4R 2 $\Rightarrow$ 8 R 2 only (I) Option

2.Sol: Suppose that set $A\cup B$ has n elements, clearly n must be 2, 3, 4, or 5. For each of these four possible values of n we can argue as follows.

There are ${}^5C_n$ ways to choose the set $A\cup B$. A can be non-empty proper subset of $A\cup B$. $A\cup B$ has 2^n subsets, but one is empty and one is all of $A\cup B$, so these are only 2^n-2 choices available for A. Thus, these ${}^5C_n\left(2^n-2\right)$ ordered pairs (A,B) with $|A\cup B|=n$.

The answer, therefore is

$$\sum_{n=2}^{5}{}^5C_n\left(2^n-2\right)=\sum_{n=2}^{5}{}^5C_n2^n-2\sum_{n=2}^{5}{}^5C_n$$

Now notice that

$$\sum_{n=2}^{5}{}^5C_n=\sum_{n=0}^{5}{}^5C_n-{}^5C_1-{}^5C_0=2^5-5-1=26,$$

and $\displaystyle\sum_{n=2}^{5}{}^5C_n2^n=\sum_{n=0}^{5}{}^5C_n2^n-{}^5C_12-{}^5C_0$

$$=\sum_{n=0}^{5}{}^5C_n2^n1^{5-n}-10-1$$

$$=(2+1)^5-11$$

$$=232$$

So the final answer is $232-52=180$.

Aliter :
If we temporarily allow A and B to be empty, we are in effect counting the ways to split X into 3 pieces, any of which may be empty. For each of the 5 elements of X we can put that element into A, into B, or into into $X(A\cup B)$. This is a 3-way choice made 5 times, so these are $3^5=243$ ways to make it. However, some of these splits leave A or B or both empty. They are the splits that put every element into B or $X(A\cup B)$, So these 2^5 of them. These are also 2^5 splits leaving B empty, so we have to substract $2\cdot 2^5=64$. However, that subtracts the one split with $A=B=\phi$ twice, so we have to add it backin. The final result is

$$243-64+1=180.$$

3.Sol: Suppose $n=19$, the set S_n has no mutliples of 19, and also, it has more than one prime. So options (a) and (b) are eliminated.

now, suppose $n=15$, multiples of 5 are 20, 25, 30. So the option (c) is also eliminated.

Therefore, finally we have S_n has 9 odd numbers and nine even numbers. Since every third odd integer is a multiple of 3, so the maximum prime numbers are 6.

4.Sol: Number of elements is B is n^2

Number of elements of type (x,x) is n.

After choosing n elements, we can choose any number of elements from remaining n^2-n elements.

i.e., $n^2-{}^nC_0+{}^{n^2-n}C_1+{}^{n^2-n}C_2+\ldots+{}^nC_{n^2-n}$

$=2^{n^2-n}$

5.Sol: Since $\cos x$ is a periodic function with period 2π. Which yields $f\left(\frac{\pi}{5}\right)=f\left(\frac{9\pi}{5}\right)$,

$$f\left(\frac{2\pi}{5}\right)=f\left(\frac{8\pi}{5}\right),f\left(\frac{3\pi}{5}\right)=f\left(\frac{7\pi}{5}\right),f\left(\frac{4\pi}{5}\right)=f\left(\frac{6\pi}{5}\right)$$

i.e., $T=f(0)-2\left[f\left(\frac{\pi}{5}\right)+f\left(\frac{\pi}{5}\right)\right]$

$$+2\left[f\left(\frac{2\pi}{5}\right)+f\left(\frac{4\pi}{5}\right)\right]-f(\pi)$$

now, we have $f(0)-f(\pi)=2(1+B+D)$

$$f\left(\frac{\pi}{5}\right)-f\left(\frac{4\pi}{5}\right)=2\left(1+B\cos\frac{3\pi}{5}+D\cos\frac{\pi}{5}\right),$$

and

$$f\left(\frac{2\pi}{5}\right)-f\left(\frac{3\pi}{5}\right)=2\left(1+B\cos\frac{6\pi}{5}+D\cos\frac{2\pi}{5}\right)$$

$\therefore\ T=2(1+B+D)$

$$-4\left[2+\left(B\cos\frac{3\pi}{5}+\cos\frac{6\pi}{5}\right)+D\left(\cos\frac{\pi}{5}+\cos\frac{2\pi}{5}\right)\right]$$

6.Sol: The function $f(x,A\cup B)$ is defined as

$$f(x,A\cup B)=\begin{cases}1, & \text{if } x\in A\cup B\\ 0, & \text{if } x\notin A\cup B\end{cases}$$

Suppose $x\in A, x\in B$, then $x\in A\cup B$. So that $f(x,A\cup B)=1$. Similarly if $x\in A, x\in B$ is and $x\notin A, x\in B$, then $x\in A\cup B$. So $f(x,A\cup B)=1$. implies that options $A,B,D.$ are eliminated.

now, if $x\notin A, x\notin B$, then $f(x,A\cup B)=0$

7.Sol: Let $I=\int\limits_0^{2012}\frac{e^{\cos(\pi\{x\})}}{e^{\cos(\pi\{x\})}+e^{-\cos(\pi\{x\})}}dx$

Since $\cos x$ is periodic, we can rewrite the given integral as

$$I=2012\int_0^1\frac{e^{\cos\pi x}}{e^{\cos\pi x}+e^{-\cos\pi x}}dx$$

using the fact

$$\int_a^b f(x)dx=\int_a^b f(a+b-x)dx,\text{ we get}$$

$$I=2012\int_0^1\frac{e^{-\cos\pi x}}{e^{-\cos\pi x}+e^{\cos\pi x}}dx$$

$\Rightarrow\quad 2I=2012\Rightarrow I=1006$

8.Sol: We know, from the definition of logarithm, $[x]>1$ and $\{x\}\neq 0$. Which yields $x\notin Z$ and $[x]>2$.

It is clear that of options (a), (b), (d) are eliminated.

So the possible interval is $(2,3)$.

Matrices and Determinants

1.Sol: Given system of linear equations has atleast two distinct solutions.

i.e., $\frac{a}{1}=\frac{1}{a+10}$

$\Rightarrow\quad a^2+10a=1$

$\therefore$ a must has two distinct values.

2.Sol: We know, a divisible by 5 contains 0, 5 in its units place. Rewrite the given determinant as

$$\begin{vmatrix}(2015-1)^{2014} & (2015)^{2015} & 2015+1)^{2016}\\ (2015+2)^{2017} & (2020-2)^{2018} & (2020-1)^{2019}\\ (2020)^{2020} & (2020+1)^{2021} & (2020+2)^{2022}\end{vmatrix}$$

$\Rightarrow$ Determinant of remainder

$$=\begin{vmatrix}(1)^{2014} & 0 & 1\\ 2^{2017} & 2^{2018} & (-1)^{2019}\\ 0 & 1^{2021} & 2^{2022}\end{vmatrix}$$

i.e., $\begin{vmatrix} 1 & 0 & 1 \\ 2^{2011} & 2^{2018} & -1 \\ 0 & 1 & 2^{2012} \end{vmatrix}$

$$= 1(2^{4040}+1)+1(2^{2017})$$

$$= ((4)^{2020}+1)+2.2^{2016}$$

$$= (5-1)^{2020}+1+2.(5-1)^{1008}$$

remainder, $(-1)^{2020}+1+2.(-1)^{1008}$

$$= 1+1+2=4$$

3.Sol: Given that $P^2 = P$

Now multiplying P^{-1} on both the side, we get

$P^{-1}P^2 = P^{-1}I$

i.e., $P = I$

now $(I+P)^n = (2P)^n = 2^n P^n$

$$= 2^n P$$

$$= P + (2^n - 1)P$$

$$= I + (2^n - 1)P$$

4.Sol: Given that $A^k = I$ and $B^k = 0$,

which yields that $\det(B) = 0$

$\therefore$ $\det(AB) = 0$

5.Sol: Given that $A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, which yields that

$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; A^3 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$ and

$A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$

Now $I + A + A^2 + A^3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

rewrite the given expression as

$\left(I + A + A^2 + A^3\right) + A^4\left(I + A + A^2 + A^3\right)$

$$+ \ldots + A^{2018}(I + A)$$

$$= (I + A)$$

$$= \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

Binomial Theorem

1.Sol: Let $I + F_2 = \left(\sqrt{2017}+44\right)^{2017}$ and

$F'_2 = \left(\sqrt{2017}-44\right)^{2017}$. It is clear that

$2017 > 44^2$, which yields that

$0 < \sqrt{2017} - 44 < 1.$

i.e., $0 < F_2' < 1.$

now $I + F - F'_2 = 2\left[{}^{2017}C_1\left(\sqrt{2017}\right)^{2016} + \ldots\right]$

which yields that $F_2 = F_2'$

$\therefore$ $F_2 = (0.911)^{2017}$

like wise $F_1 = \left(44 - \sqrt{2017}\right)^{2017}$

$$= -(0.911)^{2017}.$$

Since fractional parts always lie between 0 and 1.

That is $F_1 = 1 - (0.911)^{2017}$

now $F_1 + F_2 = 1 - (0.911)^{2017} + (0.911)^{2017}$

$$= 1$$

$\therefore$ $F_1 + F_2 = 1 - (0.911)^{2017} + (0.911)^{2017}$

$$= 1$$

$\therefore$ $F_1 + F_2$ lies between 0.9 and 1.35.

2.Sol: Given that $x = (\sqrt{50}+7)^{1/3} - (\sqrt{50}-7)^{1/3}$

Now cubing on both the sides, we get

$x^3 = \left(\sqrt{50}+7\right) - \left(\sqrt{50}-7\right)$

$-3\left(\sqrt{50}+7\right)\left(\sqrt{50}-7\right)\left(\left(\sqrt{50}+7\right)^{1/3} - \left(\sqrt{50}-7\right)^{1/3}\right)$

i.e., $x^3 = 14 - 3(1)(x)$

$\Rightarrow$ $x^3 + 3x - 14 = 0$

$\therefore$ $x = 2$ satisfies the above equation.

3.Sol: Given that

$\left(1 + x + x^2\right)^{2014} = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$+ \ldots + a_{4028} x^{4028}$$

put $x = -1$

$1 = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 \ldots$ (1)

Like wise put $x=-\omega$ and $x=-\omega^2$, we get

$$\left(1-\omega+\omega^2\right)^{2014}=a_0-a_1\omega+a_2\omega^2-a_3+a_4\omega-a_5\omega^2+a_6+\ldots \quad (2)$$

and $$\left(1-\omega^2+\omega^2\right)^{2014}=a_0-a_1\omega^2+a_2\omega-a_3+a_4\omega^2-a_5\omega+\ldots \quad (3)$$

adding eq (1), (2), (3), we get

$$1+(2\omega)^{2014}+\left(2\omega^2\right)^{2014}=3\left(a_0-a_1+a_6+\ldots\right)$$

i.e., $a_0-a_3+a_6+\ldots=\dfrac{1+2^{2014}\omega+2^{2014}\omega^2}{3}$

i.e., $A=\dfrac{1-2^{2014}}{3}=\dfrac{1+2^{2014}(-1)}{3}=\dfrac{1-2^{2104}}{3}$

$=1$

$$|A|=\frac{1-2^{2014}}{3}$$

adding $(1)+(2)\times\omega+(3)\omega^2$, we get

$$\frac{1+2^{2014}\omega^{2014}\omega+2^{2014}\left(\omega^2\right)^{2014}\omega^2}{3}$$

$=a_2-a_5+a_8+\ldots$

i.e., $$C=\frac{1+2^{2014}(\omega)^{2015}+2^{2014}(\omega)^{4030}}{3}$$

$\Rightarrow$ $$C=\frac{1+2^{2014}\left(\omega^2\right)+2^{2014}\omega}{3}$$

i.e., $$C=\frac{1-2^{2014}}{3}$$

and similarly $(1)+(2)\omega^2+(3)\omega$, gives us

$$B=\frac{1+2^{2014}\omega^{2014}\omega^2+2^{2014}\left(\omega^2\right)^{2014}\omega}{3}$$

$$=\frac{1+2^{2014}+2^{2014}}{3}$$

$$|B|=\frac{1+2^{2015}}{3}$$

Clearly, we can see that

$$|B|>|A|=|C|$$

4.Sol: Given that $\dfrac{1+x}{\left(1+x^2\right)(1-x)}$

rewrite the above expression as

$$\frac{1+x}{\left(1+x^2\right)(1-x)}=\frac{x}{1+x^2}+\frac{1}{1-x}$$

we have $\dfrac{1}{1-x}=\displaystyle\sum_{x=0}^{\infty}x^n$, and

$$\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}$$

now multiply 'x', we get

$$\frac{x}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n+1}$$

the second series does not contain power 2012 because $2n+1=2012$.

$\therefore$ The coefficient of x^{2012} is 1.

5.Sol: We know general term of a binomial expansion $(x+a)^n$ is $T_{r+1}={}^nC_rx^{n-r}a^r$.

i.e., $$T_{r+1}={}^nC_r\left(x^{1/2}\right)^{n-r}\frac{1}{\left(2x^{1/4}\right)^r}$$

$$=\frac{{}^nC_r}{2^r}x^{\frac{2n-3r}{4}}$$

also given T_1,T_2,T_3 are in A.P., that is

$$\frac{2\,{}^nC_1}{2}={}^nC_0+\frac{{}^nC_2}{2^2}$$

i.e., $$n-1=\frac{n(n-1)}{8}$$

$\Rightarrow$ $$n=8$$

also given powers of x are interger that is $\dfrac{16-3r}{4}$ must be integer. So $16-3r$ is a multiple of 4. So we have $r=0,4,8$.

Inequalities

1.Sol: Given that $P=r(4-r),q=p(4-p),$

and $r=q(4-q)$ respectively. Which yields that

$$p=4r-r^2 \quad (1)$$

$$q=4p-p^2 \quad (2)$$

$$r=4q-q^2 \quad (3)$$

adding (1), (2), (3), we get

$$p+q+r=4(p+q+r)-(p^2+q^2+r^2)$$

i.e., $(p^2-3p)+(q^2-3q)+(r^2-3r)=0 \quad (4)$

on comparing eq (4) both sides, we get

$$p^2-3p=0 \Rightarrow p=0,3.$$

like wise $q=0,3$ and $r=0,3$.

maximum value of p,q,r are 3, 3, 3 respectively.

$\therefore$ Maximum value of $p+q+r$ is $3+3+3=9$.

2.Sol: Given that (i) $a+b<c+d$

(ii) $b+c<d+e$

(iii) $c+d<e+a$

(iv) $d+e<a+b$

From (i) and (iii), we get

$$a+b<e+a$$

i.e., $b<e$

From (ii) and (iv), we get

$$b+c<a+b$$

i.e., $c>a$

Subtracting equation (i) and (ii), we get

$$a-c<c-e$$

i.e., $c>e$

Subtracting equation (i) and (iv), we get

$$(a-e)+(b-d)<(c-a)+(d-b)$$

i.e., $d>b$

Thus we can conclude that 'a' is greatest and 'b' is least.

3.Sol: By Cauchy Schwarz inequality, we have

$$\left\{\int_a^b f^2(x)g(x)dx\right\}^2 \le \int_a^b (f(x))^2 dx \int_a^b (g(x))^2 dx$$

Here $g(x)=1$ and equality holds only when

$$\frac{f(x)}{g(x)}=\lambda$$

So, $f(x)$ is constant

4.Sol: Let $x=\frac{3}{2}, y=\frac{5}{3}$

now $[x+y]=\left[\frac{3}{2}+\frac{5}{3}\right]=\left[\frac{19}{6}\right]>\left[\frac{3}{2}\right]+\left[\frac{5}{3}\right]$

so $[x+y]\le[x][y]$ is false

like wise, $[x,y]=\left[\frac{3}{2}\frac{5}{3}\right]=\left[\frac{5}{2}\right]\ge\left[\frac{3}{2}\right]\left[\frac{5}{3}\right]$

so $[xy]\le[x][y]$ is also false

now, put $x=\frac{7}{2}$, gives $\left[2^{\frac{7}{2}}\right]=\left[\sqrt{128}\right]$

$\Rightarrow \quad \left[\sqrt{128}\right]\ge 2^3$

i.e., $\quad \left[2^{\frac{7}{2}}\right]\ge 2^{\left[\frac{7}{2}\right]}$

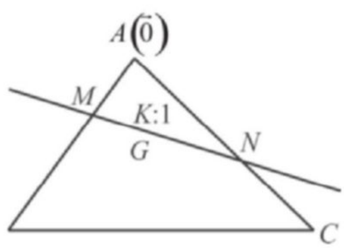
so $\quad \left[2^x\right]\le 2^{[x]}$ is false

and finally $\left[\frac{x}{y}=\left[\frac{\frac{3}{2}}{\frac{5}{3}}\right]\right]=\left[\frac{9}{10}\right]\le\frac{1}{1}=\frac{\left[\frac{3}{2}\right]}{\left[\frac{5}{3}\right]}$

$\therefore \left[\frac{x}{y}\right]\le\frac{[x]}{[y]}$ is True.

5.Sol: By taking different values of a and k. option (B) is possible.

6.Sol: Let $\overrightarrow{AB}=\vec{b}, \overrightarrow{AC}=\vec{c}$, $\overrightarrow{AM}=\lambda\vec{b}$ and $\overrightarrow{AN}=m\vec{c}$



Let G divides MN in the ratio $k:1$

So $\quad \frac{k\mu\vec{c}+\lambda\vec{b}}{k+1}=\frac{\vec{b}+\vec{c}}{3}$

$\Rightarrow \quad \frac{k\mu}{k+1}=\frac{1}{3}$ and $\frac{\lambda}{k+1}=\frac{1}{3}$

$\Rightarrow \quad k = \frac{\lambda}{\mu} \Rightarrow \frac{1}{\lambda} + \frac{1}{\mu} = 3$

Using $AM \geq GM$ in equality, we get

$$\frac{\frac{1}{\lambda}+\frac{1}{\mu}}{2} \geq \frac{1}{\sqrt{\lambda\mu}} \Rightarrow \left(\frac{2}{3}\right)^2 \leq \lambda\mu \qquad (1)$$

Now, $\frac{\text{area of } \Delta AMN}{\text{area of } ABC} = \frac{\frac{1}{2}\lambda\mu|\vec{b}\times\vec{c}|}{\frac{1}{2}|\vec{b}\times\vec{c}|}$

$= \lambda\mu$

using $\frac{1}{\lambda}+\frac{1}{\mu}$ is equal to 3. we have the ratio $\frac{\lambda^2}{3\lambda-1}, \lambda \in [0,1]$.

$\therefore$ maximum value of ratio $\frac{\lambda^2}{3\lambda-1}$ attain when $\lambda = 1$ using deriviative but λ is not 1 because M is an interior point.

so $\quad \frac{4}{9} \leq r < \frac{1}{2}$

7.Sol: Given that $3^x + 5^x - 9^x + 15^x - 25^x$

we have $A.M - G.M$ inequality

now $\quad 9^x + 25^x \geq 2(15)^x$

i.e., $\quad -(9^x + 25^x) \leq -2(15)^x$

now, the given expression is written as

$$3^x + 5^x - 9^x + 15^x - 25^x \leq 3^x + 5^x - 15^x$$

$$0 < 1 - (3^x - 1)(5^x - 1)$$

Clearly $\quad 0 < M < 2$.

8.Sol: We have

$$2^2 + 4^2 + 6^2 + \ldots + (2n)^2 = \frac{2n(2n+1)(2n-1)}{6}$$

and $\quad 1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2$

$$= \frac{n}{3}(2n-1)(2n+1)$$

given that $\frac{2^2 + 4^2 + 6^2 + \ldots + (2n)^2}{1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2} < 1.01$

i.e., $\quad \frac{\frac{2n(2n+1)(2n+2)}{6}}{\frac{n(2n-1)(2n+1)}{3}} < 0$

i.e., $\quad \frac{2n+2}{2n-1} < 1.01$

$\Rightarrow \quad \frac{3}{2n-1} < 0.01$

i.e., $\quad 300 < 2n - 1$

$\therefore \quad n > \frac{301}{2}$

Complex Numbers

1.Sol: Given that $A_r = e^{i\pi rn}, n \in N$.

now, we have $A_1 = e^{i\pi n}$

$= \cos n\pi$

$\therefore \quad A_1 = \{-1, 1\}$

Hence A_1 is a finite set.

like wise $A_{\frac{1}{\pi}} = e^{i\pi\left(\frac{1}{\pi}\right)n} = e^{in}$

$= \cos n + i \sin n$, where n is in radians.

$\therefore$ Since period of $\cos n + i \sin n$, is $\frac{2\pi}{n} \notin I$, so neither it repeats itself nor it has integer period.

Hence $A_{\frac{1}{\pi}}$ have infinite number of elements.

now $A_{0.3} = e^{i\left(\frac{3n\pi}{10}\right)} = \cos\left(\frac{3n\pi}{10}\right) + i \sin\left(\frac{3n\pi}{20}\right)$

Since period is $\frac{20}{3n}$, which has a common multiple with a and 20, for all n.

$\therefore \quad A_{0.3}$ is a periodic with period 20, for all n.

Hence $A_{0.3}$ contains 20 distinct elements.

2.Sol: Given that $|a|<1$ and $w=\left(\dfrac{z-a}{1-\bar{a}z}\right)$

$\Rightarrow \quad w+a=z(1+\bar{a}w)$

i.e., $\quad z=\dfrac{w+a}{1+\bar{a}w}$

also given that $|z|<1$

i.e., $\quad \left|\dfrac{w+a}{1+\bar{a}w}\right|<1$

$\Rightarrow \quad (w+a)^2<(1+\bar{a}w)^2$

$\Rightarrow \quad (w+a)(\bar{w}+\bar{a})<(1+\bar{a}w)(1+a\bar{w})$

i.e., $w\bar{w}+w\bar{a}+a\bar{w}+a\bar{a}<1+\bar{a}w+a\bar{w}+a\bar{a}w\bar{w}$

$\Rightarrow \quad a\bar{a}\,w\bar{w}-w\bar{w}-a\bar{a}+1>0$

$\Rightarrow \quad |a|^2|w|^2-|w|^2-|a|^2+1>0$

i.e., $\quad \left(|a|^2-1\right)\left(|w|^2-1\right)>0$

since $|a|<1$ i.e., $|a|^2-1<0$ so $|w|^2-1<0$

i.e., $|w|<1$ and $|z|<1$

3.Sol: Given that $\left|z^3+\dfrac{1}{z^3}\right|\le 2.$ But we have from Triangular is equality that

$$\left|z^3+\frac{1}{z^3}\right|\le |z|^3+\frac{1}{|z|^3}$$

Which yields that $|z|^3+\dfrac{1}{|z|^3}=2.$

Therefore using $A.M-G.M$ inequality we have $|z|=1$

now $\quad \left|z+\dfrac{1}{z}\right|\le |z|+\dfrac{1}{|z|}=2.$

$\therefore$ maximum value is 2.

4.Sol: $\sum_{k=0}^{n} {}^nC_k\omega^k={}^nC_0+{}^nC_1\omega+\ldots+{}^nC_0\omega^n$

$$=(1+\omega)^n=(-\omega^2)^n=(-1)^n\,\omega^2 n$$

$\therefore \quad \left|e^{(-1)^n\omega^{2n}}\right|=\left|e^{(-\omega^2)^n}\right|$

$$=\left|e^{\left(-\cos\frac{4\pi}{3}-i\sin\frac{4\pi}{3}\right)^n}\right|$$

$$=\left|e^{\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}}\right|$$

$$=\left|e^{\cos\frac{n\pi}{3}}\right|$$

when n is a multiple of 3, we have $e,\dfrac{1}{e}$ and when n is not a multiple of 3, we have $e^{\frac{1}{2}},e^{\frac{-1}{2}}$

$$=\left\{e^1,e^{1/2},e^{-1/2},e^{-1}\right\}$$

Four values.

5.Sol: Given that $|a+bw+cw^2|$

Rewrite the given expression as

$|a-c+(b-c)w|$, for maximum value taking

$a=1,c=-1,b=1$

$|a+bw+cw^2|=|2+2w|=2|w^2|=2$

6.Sol: Let $S=i+2i^2+3i^3+\ldots+ni^3$

Since it in $A\cdot G\cdot P$, we have

$$S=\frac{1-i^n}{-2i}-\frac{ni^{n+1}}{1-i}$$

also given that $|S|=18\sqrt{2}$

i.e., $\quad \left|\dfrac{1-i^n}{-2i}-\dfrac{ni^{n+1}}{1-i}\right|=18\sqrt{2}$

we have $\left|\dfrac{1-i^n}{-2i}\right|=\dfrac{1}{\sqrt{2}}$ or 0 and $\left|\dfrac{ni^{n+1}}{1-i}\right|=\dfrac{n}{\sqrt{2}}$

$\therefore \quad \left|-\dfrac{n}{\sqrt{2}}=18\sqrt{2}\right|$

i.e., $\quad n=36.$

7.Sol: Given that $1,\omega,\omega^2$ are cube roots of unity.

i.e., $\omega = \frac{-1+\sqrt{3}i}{2}, \omega^2 = \frac{-1-\sqrt{3}i}{2}$

now $2\omega^2 = -1-\sqrt{3}i, 3+4\omega = 1+2\sqrt{3}i,$

$3+4\omega^2 = 1-2\sqrt{3}i$ and $5-\omega-\omega^2 = 6$

Since, it has 3 imaginary roots, out of which two are conjugate pairs to each other. So the least possible degree of the polynomial is 5.

Three Dimensional Geometry

1.Sol: Sphere $x^2+y^2+z^2-4x-6x-12x+48=0$

Centre is at $(2,3,6)$ and radius

$= \sqrt{4+9+36-48} = 1$

Now distance between centre and origin is

$\sqrt{4+9+36} = 7$

and shortest distance is $7-1=6$ (Origin lies outside the sphere).

Coordinate System

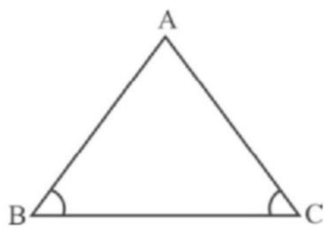
1.Sol: It has 3 cases and those are defined as

(i) $\angle B = \angle C$

(ii) $\angle A = \angle C$ and

(iii) $\angle A = \angle B$

Case (i) :

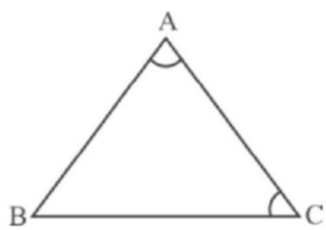


when $\angle B = \angle C$

locus of A is perpendicular bisector of BC

So it is a striaght line

Like wise case (ii)



we have $\angle A = \angle C$

BC fixed $B(a,0), C(0,a)$ and we have

$BC = AB$

So, $(x-a)^2 + y^2 = 2a^2$

Circle

Simillarly case (iii)

we have $\angle A = \angle B$

$\Rightarrow$ $AC = BC$

$\Rightarrow$ $\sqrt{h^2+(k-a)^2} = \sqrt{2a^2}$

$\Rightarrow$ $x^2+(y-a)^2 = 2a^2$

so case (iii) in also circle

So union of two circle and a line.

2.Sol: Given that $AB = \sqrt{(a_1-b_1)^2+(a_2-b_2)^2}$

we need to verify the numbers given options, which can be written as sum of two square number. That

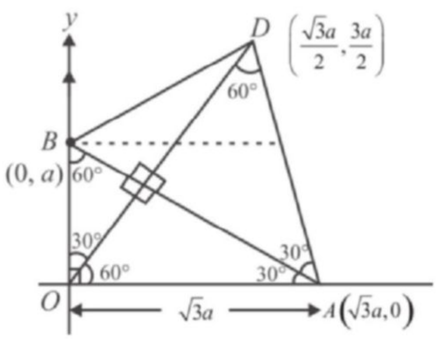
$\sqrt{65} = \sqrt{64+1},$

$\sqrt{74} = \sqrt{49+25},$

$\sqrt{97} = \sqrt{81+16}.$

We can not decompose 83 into sum of two square numbers.

3.Sol:

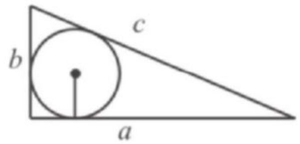


$\therefore$ slope of DB is $\frac{\frac{3a}{2}-a}{\frac{\sqrt{3}}{2}a-0} = \frac{1}{\sqrt{3}}$

$M_{BD} = \frac{1}{\sqrt{3}}$

4.Sol: We know the radius of incircle is

$$\frac{ab}{a+b+c} \text{ or } \frac{a+b-c}{2}$$



As one of the side is 12, Let the other side be x and then hypotenuse is $\sqrt{144+x^2}$ and radius of in circle is

$$r(x)=\frac{12+x-\sqrt{144+x^2}}{2}$$

$$=6+\frac{1}{2}\left(x-\sqrt{144+x^2}\right)$$

Given that all sides are integers, if h is hypotenuse and we have one side as x, then

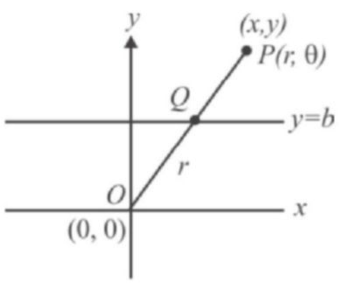
$144=h^2-x^2=(h+x)(h-x)$ and sum of possible factors of 144 has to be even as h and x are integers.

possible solutions are $2\times72, 4\times36, 6\times24$ and 12×12. The latter is not admissible as it results in one side to be 0. The first four results give values of (h,x) as $(37,35),(20,16),(15,9)$, and $(13,5)$.

And for these we get $r=\{2,3,4,5\}$.

Hence largest incircle possible is of radius 5 and dimension of triangle are (12, 35, 37).

5.Sol:



equation of OP is $y=x\tan\theta$ and

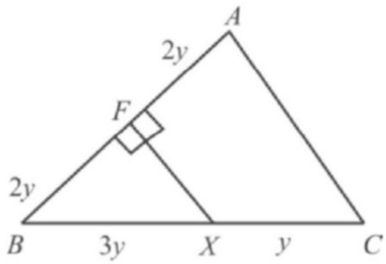
Point Q is $(b\cot\theta, b)$

$\therefore$ point P is $y=b\pm d\sin\theta$

i.e., $\quad r\sin\theta=b\pm d\sin\theta$

$\Rightarrow \quad (r\pm d)\sin\theta=b$

6.Sol:



From ΔFBX, we have

$\Rightarrow \quad \cos B=\frac{2}{3}$

and from ΔABC, we have

$$\cos B=\frac{16y^2+16y^2-AC^2}{2\cdot4y\cdot4y}$$

i.e., $\quad \frac{2}{3}=\frac{32y^2-AC^2}{32y^2}$

$\Rightarrow \quad 64y^2+16y^2-3AC^2$

$\Rightarrow \quad 3AC^2=32y^2$

$\Rightarrow \quad AC^2=\frac{32}{3}y^2$

$\Rightarrow \quad AC=\frac{4\sqrt{2}}{\sqrt{3}}y$

Now, $\quad \frac{BC}{AC}=\frac{4y}{\left(\frac{4\sqrt{2}y}{\sqrt{3}}\right)}=\frac{\sqrt{3}}{\sqrt{2}}$

7.Sol: Given that $ax+9y=5$ and $4x+by=3$ are parallel. That is

$$\frac{a}{4}=\frac{9}{b}$$

$\Rightarrow \quad ab=36$

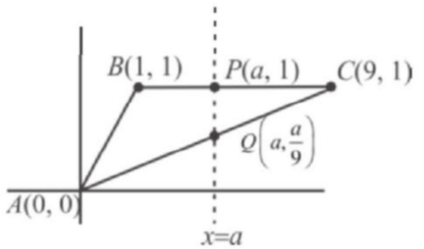
we have $A.M-G.M$ inequality

i.e., $\quad \frac{a+b}{2}\geq\sqrt{ab}$

$$\Rightarrow \quad a+b \geq 2\sqrt{36} = 12$$

$\therefore$ The least possible value of $a+b$ is 12.

8.Sol: Given that, area of ΔPQC is $\frac{1}{2}$ of the area of ΔABC.



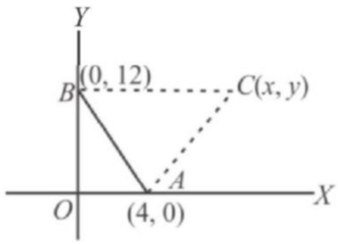
i.e., $\frac{1}{2}(9-a)\left(1-\frac{a}{9}\right) = \frac{1}{2}\frac{1}{2}(8\times 1)$

$\Rightarrow \quad (9-a)^2 = 4\times 9$

i.e., $\quad 9-a = \pm 6$

$\therefore \quad a = 3 \ \{\because 0<a<9\}$

10.Sol:



we have Area of a triangle

i.e., $\frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & 0 & 12 \\ 1 & 4 & 0 \end{vmatrix} = \pm 18$

$\Rightarrow \quad 1(-48) - x(-12) + y(4) = \pm 36$

$\Rightarrow \quad 12x + 4y - 48 = \pm 36$

$\Rightarrow \quad 3x + y - 12 = \pm 9$

i.e., $\quad (3x+y-12)^2 = 81$

11.Sol: We have $A+B+C = 180°$

given that $3\sin A + 4\cos B = 6 \quad (1)$

$4\sin B + 3\cos A = 1 \quad (2)$

Squaring (1) and (2), then add (1) and (2), we get

$$(3\sin A + 4\cos B)^2 + (4\sin B + 3\cos A)^2 = 6^2 + 1^2$$

i.e., $9(\sin^2 A + \cos^2 A) + 16(\cos^2 B + \sin^2 B)$

$+24(\sin A\cos B + \cos A\sin B) = 37$

$\Rightarrow \quad 9 + 16 + 24\sin(A+B) = 37$

$\Rightarrow \quad 24\sin(A+B) = 12$

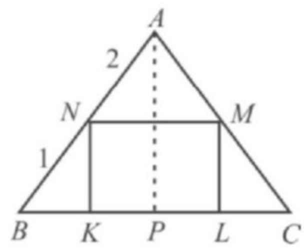
i.e., $\quad \sin(A+B) = \frac{1}{2}$

$\Rightarrow \quad \sin(180-C) = \frac{1}{2} \ \{\because A+B = 180-C\}$

i.e., $\quad \sin C = \frac{1}{2}$

$\therefore \quad C = 30°$

12.Sol: Given that Area of ΔBKN is 6 and $\frac{AN}{NB} = 2.$



we have : If two triangles are similar, than the ratio of the area of both triangles is proportional to the square of the ratio of their corresponding sides.

Since $\Delta BKN \sim \Delta APB$, we have

$$\frac{\Delta BKN}{\Delta APB} = \left(\frac{BN}{AB}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

$\Rightarrow \quad \Delta APB = 9\Delta BKN$

$= 9(6)$

$= 54$

and we know area of ΔABC is twice the area of ΔAPB

i.e., $\Delta ABC = 2(\Delta APB)$

$= 2(54)$

$= 108$

Previous year JEE MAIN Questions

CIRCLES

[ONLINE QUESTIONS]

1. The equation $\operatorname{Im}\left(\frac{iz-2}{z-i}\right)+1=0, z\in C, z\neq 1$ represents a part of a circle having radius equal to : **[2017]**

(a) 2 (b) 1 (c) $\frac{3}{4}$ (d) $\frac{1}{2}$

2. A line drawn through the point $P(4,7)$ cuts the circle $x^2+y^2=9$ at the points A and B. Then $PA\cdot PB$ is equal to : **[2017]**

(a) 53 (b) 56 (c) 74 (d) 65

3. The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is $60°$. If the area of the quadrilateral is $4\sqrt{3}$, then the perimeter of the quadrilateral is : **[2017]**

(a) $12\cdot5$ (b) $13\cdot2$ (c) 12 (d) 13

4. Let $z\in C$, the set of complex numbers. Then the equation, $2|z+3i|-|z-i|=0$ represents : **[2017]**

(a) A circle with radius $\frac{8}{3}$

(b) A circle with diameter $\frac{10}{3}$

(c) An ellipse with length of major axis $\frac{16}{3}$

(d) An ellipse with length of minor axis $\frac{16}{9}$

5. If a point P has co-ordinates $(0,-2)$ and Q is any point on the circle, $x^2+y^2-5x-y+5=0$, then the maximum value of $(PQ)^2$ is : **[2017]**

(a) $\frac{25+\sqrt{6}}{2}$ (b) $14+5\sqrt{3}$

(c) $\frac{47+10\sqrt{6}}{2}$ (d) $8+5\sqrt{3}$

6. If two parallel chords of a circle, having diameter 4 units, lie on the opposite sides of the centre and subtend angles $\cos^{-1}\left(\frac{1}{7}\right)$ and $\sec^{-1}(7)$ at the centre respectively, then the distance between these chords, is : **[2017]**

(a) $\frac{4}{\sqrt{7}}$ (b) $\frac{8}{\sqrt{7}}$ (c) $\frac{8}{7}$ (d) $\frac{16}{7}$

7. Equation of the tangent to the circle, at the point $(1,-1)$ whose centre is the point of intersection of the straight lines $x-y=1$ and $2x+y=3$ is :

[2016]

(a) $x+4y+3=0$ (b) $3x-y-4=0$

(c) $x-3y-4=0$ (d) $4x+y-3=0$

8. A circle passes through $(-2,4)$ and touches the y-axis at $(0,2)$. Which one of the following equations can represent a diameter of this circle [2016]

(a) $2x-3y+10=0$ (b) $3x+4y-3=0$

(c) $4x+5y-6=0$ (d) $5x+2y+4=0$

9. If the incentre of an equilateral triangle is (1, 1) and the equation of its one side is $3x+4y+3=0$, then the equation of the circumcircle of this triangle is : [2015]

(a) $x^2+y^2-2x-2y-14=0$

(b) $x^2+y^2-2x-2y-2=0$

(c) $x^2+y^2\_2x-2y+2=0$

(d) $x^2+y^2-2x-2y-7=0$

10. If a circle passing through the point $(-1,0)$ touches y-axis at (0, 2), then the length of the chord of the circle along the x-axis is : [2015]

(a) $\frac{3}{2}$ (b) 3 (c) $\frac{5}{2}$ (d) 5

11. Let the tangents drawn to the circle $x^2+y^2=16$ from the point $P(0,h)$ meet the x-axis at point A and B. If the area of ΔAPB is minimum, then h is equal to : [2015]

(a) $4\sqrt{2}$ (b) $3\sqrt{3}$ (c) $3\sqrt{2}$ (d) $4\sqrt{3}$

12. If $y+3x=0$ is the equation of a chord of the circle, $x^2+y^2-30x=0$, then the equation of the circle with this chord as diameter is : [2015]

(a) $x^2+y^2+3x+9y=0$

(b) $x^2+y^2+3x-9y=0$

(c) $x^2+y^2-3x-9y=0$

(d) $x^2+y^2-3x+9y=0$

13. The largest value of r for which the region represented by the set $\{\omega\in C|\omega-4-i|\le r\}$ is contained in the region represented by the set $(z\in c/|z-1|\le|z+i|)$, is equal to [2015]

(a) $\frac{5}{2}\sqrt{2}$ (b) $2\sqrt{2}$ (c) $\frac{3}{2}\sqrt{2}$ (d) $\sqrt{17}$

14. The equation of circle described on the chord $3x+y+5=0$ of the circle $x^2+y^2=16$ as diameter is : [2014]

(a) $x^2+y^2+3x+y-11=0$

(b) $x^2+y^2+3x+y+1=0$

(c) $x^2+y^2+3x+y-2=0$

(d) $x^2+y^2+3x+y-22=0$

15. For the two circles $x^2+y^2=16$ and $x^2+y^2-2y=0$, there is/are [2014]

(a) One pair of there is/are

(b) Two pair of common tangents

(c) Three pair of common tangents

(d) No common tangent

16. The set of all real values of λ for which exactly two common tangents can be drawn to the circles $x^2+y^2-4x-4y+6=0$ and $x^2+y^2-10x-10y+\lambda=0$ is the interval : [2014]

(a) $(12,32)$ (b) $(18,42)$

(c) $(12,24)$ (d) $(18,48)$

ANSWER KEY

1. c	2. b	3. c	4. a	5. b
6. b	7. a	8. a	9. a	10. b
11. a	12. d	13. a	14. a	15. d
16. b				

HINTS & SOLUTIONS

1.Sol: Let $z=x+y$

$\Rightarrow\quad z-i=x+i(y-1)$

and $iz-2=i(x+iy)-2$

$=ix-y-2$

$=(-y-2)+ix$

now $\dfrac{iz-2}{z-i}=\dfrac{(-y-2)+ix}{x+i(y-1)}$

$$=\frac{(-y-2)x+ix^2-i(y-1)(-y-2)+x(y-1)}{x^2+(y-1)^2}$$

$$\Rightarrow \operatorname{Im}\left(\frac{iz-2}{z-i}\right)=\frac{x^2-(y-1)(-y-2)}{x^2+(y-1)^2}=-1$$

$$x^2+(y^2+y-2)=-x^2-(y-1)^2$$

$$2x^2+(y^2+y-2)+y^2-2y+1=0$$

$$2x^2+2y^2-y-1=0$$

$$\Rightarrow \qquad x^2+y^2-\frac{1}{2}y-\frac{1}{2}=0$$

$\therefore$ radius is $\sqrt{g^2+f^2-c}$.

$$=\sqrt{0+\left(\frac{1}{4}\right)^2+\frac{1}{2}}$$

$$\Rightarrow \sqrt{\frac{9}{16}}=\frac{3}{4}$$

2.Sol: We know relation between secant and tangent of a circle

i.e., $PA\cdot PB=PT^2$

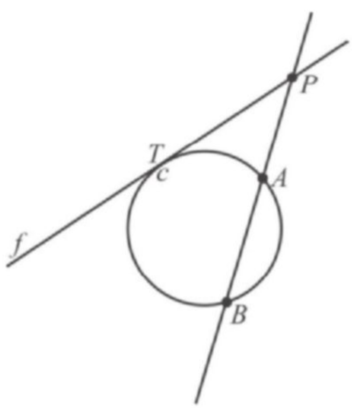
We have

$$PT=\sqrt{S_{11}}$$

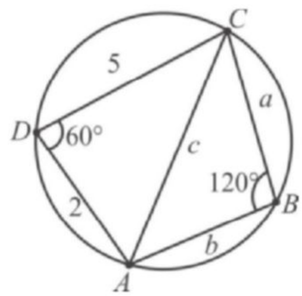
$$\therefore \quad PA\cdot PB=\left(\sqrt{S_{11}}\right)^2=S_{11}$$

$$=4^2+7^2-9$$

$$=56$$



3.Sol: Given that $ABCD$ is a cyclic quadrilateral.



Let AD = 2, CD = 5 and $\angle ADC=60°$

$\Rightarrow \quad \angle ABC=180\angle ADC=120°$

Now form ΔADC, we have

$$\cos 60°=\frac{4+25-c^2}{2(2)(5)}$$

$$\frac{1}{2}(20)=29-c^2$$

i.e., $\quad c^2=19$

$\Rightarrow \quad c=\sqrt{19}$

From ΔABC, we have

$$\cos 120°=\frac{a^2+b^2-c^2}{2ab}$$

$$\frac{-1}{2}=\frac{a^2+b^2-19}{2ab}$$

$\Rightarrow \quad -ab=a^2+b^2-19$

$\Rightarrow \quad a^2+b^2+ab=19 \qquad (1)$

also given area of quadrilateral $ABCD$ is $4\sqrt{3}$

i.e., Area of $\Delta ADC+$ Area of ΔABC

$$\Rightarrow \quad \frac{1}{2}\times 2\times 5\sin 60+\frac{1}{2}ab\sin 120°=4\sqrt{3}$$

$$\Rightarrow \quad \frac{5\sqrt{3}}{2}+\frac{\sqrt{3}}{4}ab=4\sqrt{3}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{4}ab=4-\frac{5}{2}$$

$$\Rightarrow ab=\frac{3}{2}\times 4=6$$

$$\Rightarrow \quad a^2+b^2=13$$

now $a=2, b=3$

$\therefore$ Perimeter is sum of all sides

i.e., $2+5+2+3=12$.

4.Sol: Let $\quad z=x+iy$

$$\Rightarrow \quad 2|x+i(y+3)|=|x+i(y-1)|$$

$$\Rightarrow \quad 2\sqrt{x^2+(y+3)^2}=\sqrt{x^2+(y-1)^2}$$

$$\Rightarrow \quad 4x^2+4(y+3)^2=x^2+(y-1)^2$$

$$\Rightarrow \quad x^2+y^2+\frac{26}{3}y+\frac{35}{3}=0$$

$$\Rightarrow \quad r=\sqrt{0+\frac{169}{9}-\frac{35}{3}}$$

$$\Rightarrow \quad r=\sqrt{\frac{65}{9}}=\frac{8}{3}$$

5.Sol: Given that $x^2+y^2-5x-y+5=0$

$$\Rightarrow \quad (x-5/2)^2-\frac{25}{4}+(y-1/2)^2-1/4=0$$

$$\Rightarrow \quad (x-5/2)^2+(y-12)^2=3/2$$

on circle

$$\left[Q\equiv\left(5/2+\sqrt{3/2}\cos\theta,\frac{1}{2}+\sqrt{3/2}\sin\theta\right)\right]$$

$$\Rightarrow PQ^2=\left(\frac{5}{2}+\sqrt{3/2}\cos\theta\right)^2+\left(\frac{5}{2}+\sqrt{3/2}\sin\theta\right)^2$$

$$\Rightarrow \quad PQ^2=\frac{25}{2}+\frac{3}{2}+5\sqrt{3/2}\,(\cos\theta+\sin\theta)$$

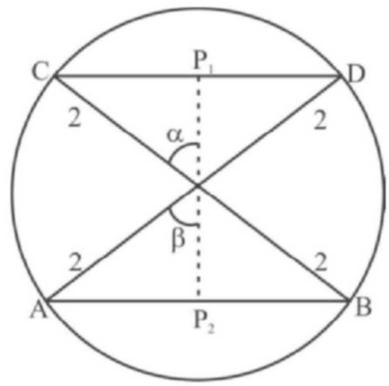
$$=14+5\sqrt{3/2}\,(\cos\theta+\sin\theta)$$

We have $-\sqrt{2}\le\cos\theta+\sin\theta\le\sqrt{2}$

$\therefore$ Maximum value of PQ^2

$$=14+5\sqrt{3/2}\times\sqrt{2}=14+5\sqrt{3}$$

6.Sol: Let $2\alpha=\cos^{-1}\left(\frac{1}{7}\right)$ and $2\beta=\sec^{-1}(7)$



$$\Rightarrow \quad \cos 2\alpha=\frac{1}{7} \text{ and } \sec 2\beta=7$$

i.e., $2\cos^2\alpha-1=\frac{1}{7}$ and $\frac{1}{2\cos^2-1}=7$

$$\Rightarrow \quad 2\cos^2\alpha=\frac{8}{7} \text{ and } 2\cos^2\beta=\frac{8}{7}$$

$$\therefore \quad \cos^2\alpha=\frac{4}{7} \text{ and } \cos^2\beta=\frac{4}{7}$$

i.e., $\cos\alpha=\frac{2}{\sqrt{7}}$ and $\cos\beta=\frac{2}{\sqrt{7}}$

$$\therefore \quad P_1P_2=r\cos\alpha+r\cos\beta$$

$$=\left(\frac{4}{\sqrt{7}}+\frac{4}{\sqrt{7}}\right)=\frac{8}{\sqrt{7}}$$

7.Sol: Equation of any line through the intersection of $x-y=1$ and $2x+y=3$ is

$$\lambda(x-y=1)+(2x+y=3)=0 \qquad (1)$$

Let it pass through (1, -1) when it becomes a radial line.

$$\lambda(1-(-1)+(2(1)+(-1)-3)=0$$

$$\lambda-2=0 \quad \to\lambda=2$$

Plugging it in (1) we get $4x-y-5=0$ tangent at (1, -1) is perpendicular to it. To get the equation exchange the coefficients of x and y and change the sign between them and write them on L.H.S. of the equation and write on R.H.S. its value at (1, -1)

8.Sol: Required circle is

$$(x-0)^2+(y-2)^2+\lambda(x)=0$$

also given it passes through $(-2,4)$

That is $4+4-2\lambda=0$

$\lambda=4$

Therefore equation of required circle is

$x^2+y^2+4x-4y+4=0$ centre of the circle is (-2, 2). Which satisfies the equation $2x-3y+10=0$.

9.Sol: We have, the relation between in radius (r) and circum radius (R) of an equilateral triangle

i.e., $R=2r$

We have in radius (r)

$$r=\frac{|3(1)+4(1)+3|}{\sqrt{3^2+4^2}}=\frac{10}{2}=2$$

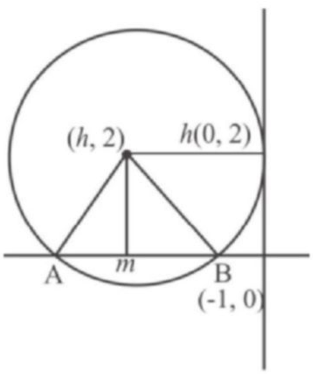
$\therefore \quad R=4$

So equation of required circle is

$$(x-1)^2+(y-1)^2=16$$

$$\Rightarrow \quad x^2+y^2-2x-2y-14=0$$

10.Sol: Let 'h' be the radius of the circle and since circle touches y-axis at (0, 2) therefore centre $=(h,2)$



Now, equation of circle is $(h+1)^2+2^2=h^2$

$$\Rightarrow \quad 2h+5=0$$

$$h=-\frac{5}{2}$$

From the figure, it is clear that AB is the chord along x-axis.

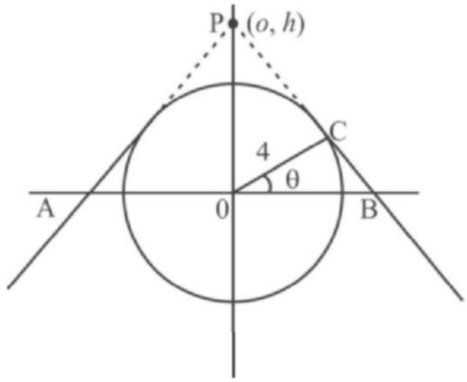
$$\therefore \quad AB=2(AM)=2\sqrt{\frac{25}{4}-4}=2\left(\frac{3}{2}\right)=3$$

11.Sol: From ΔOCP, we have

$$OP=\frac{4}{\sin\theta}$$

Similarly, from ΔOCB,

We have $\cos\theta=\frac{OC}{OB} \Rightarrow \quad OB=\frac{4}{\cos\theta}$



$$\text{Area}=OP\times OB=\frac{16}{\sin\theta\cos\theta}=\frac{32}{\sin 2\theta}$$

Area is minimum when $\sin 2\theta$ is maximum

i.e., $\sin 2\theta=1$

$\Rightarrow \quad \theta=45°$

$$\therefore \quad h=\frac{4}{\sin 45°}=4\sqrt{2}$$

12.Sol: Given that $y+3x=0$ is the equation of a chord of the circle then

$$y=-3x \qquad (1)$$

$$(x^2)+(-3x)^2-30x=0$$

$$\Rightarrow \quad 10x^2-30x=0$$

$$\Rightarrow \quad 10x(x-3)=0$$

$$\Rightarrow \quad x=0,3$$

ends of chord are (0, 0) and (3, 9)

so the equation of the circle is

$$(x-3)(x-0)+(y+9)(y-0)=0$$

$$\Rightarrow \quad x^2-3x+y^2+9y=0$$

i.e., $\quad x^2+y^2-3x+9y=0$

13.Sol: $|w-4-i|\le r$

$\Rightarrow$ centre of cirle is (4, 1) and radius 'r'

now $|z-1| \le |z+i|$

$\Rightarrow$ straight line $y=-x$

$\therefore$ Maximum $r=\dfrac{4+1}{\sqrt{1+1}}=\dfrac{5}{\sqrt{2}}=\dfrac{5\sqrt{2}}{2}$

14.Sol: Given circle is $x^2+y^2-16=0$

Equation of chord say AB of given circle is

$$3x+y+5=0$$

Equation of required circle is

$$x^2+y^2-16+\lambda(3x+y+5)=0$$

$\Rightarrow$ $x^2+y^2+(3\lambda)x+(\lambda)y+5\lambda-16=0$

Centre $C=\left(\dfrac{-3\lambda}{2},\dfrac{-\lambda}{2}\right)$

If line AB is the diameter of circle (1), then

$C=\left(\dfrac{-3\lambda}{2},\dfrac{-\lambda}{2}\right)$ will lie on line AB

i.e., $\quad 3\left(\dfrac{-3\lambda}{2}\right)+\left(\dfrac{-\lambda}{2}\right)+5=0$

$\Rightarrow \quad -\dfrac{9\lambda-\lambda}{2}+5=0 \Rightarrow \lambda=1$

Hence, required equation of circle is

$$x^2+y^2+3x+y+5-16=0$$

$\Rightarrow \quad x^2+y^2+3x+y-11=0$

15.Sol: Let, $x^2+y^2=16$ or $x^2+y^2=4^2$

radius of circle $r_1=4$, centre $C_1(0,0)$

we have, $x^2+(y^2-2y+1)-1=0$

$\Rightarrow$ $x^2+y^2-2y=0$ or $x^2+(y-1)^2=1^2$

Radius 1, centre $C_2(0,1)$

$$|C_1C_2|=1$$

$$|r_2-r_1|=|4-1|=3$$

$$|C_1C_2|<|r_2-r_1|$$

16.Sol: The equations of the circles are

$$x^2+y^2-10x-10y+\lambda=0 \qquad (1)$$

and $x^2+y^2-4x-4y+6=0 \qquad (2)$

$C_1=$ centre of (1) $=(5,5)$

$C_2=$ centre of (2) $=(2,2)$

$d=$ distance between centers

$=C_1C_2=\sqrt{9+9}=\sqrt{18}$

$r_1=\sqrt{50-\lambda}, r_2=\sqrt{2}$

For exactly two common tangents we have

$$r_1-r_2<C_1C_2<r_1+r_2$$

$\Rightarrow$ $\sqrt{50-\lambda}-\sqrt{2}<3\sqrt{2}<\sqrt{50-\lambda}+\sqrt{2}$

$\Rightarrow$ $50-\lambda<32$ or $8<50-\lambda$

$\Rightarrow$ $\lambda>18$ or $\lambda<42$

Required interval is $(18,42)$

MOCK TEST PAPER

JEE MAIN - 8

2018 2019

1. If $P(A) = P(B)$, then
   (a) $A \subset B$ (b) $B \subset A$
   (c) $A = B$ (d) All of these

2. A = {words which have the letter a in them}; B = {words which have the letter b in them}. What is $(A \mid B) \cup (B \mid A)$?
   (a) Words which have both the letters a and b in them
   (b) Words which have the letters a or b in them, but not both
   (c) Words which have the letters a or b in them
   (d) Words which do not have the letters a or b in them

3. The function f is continuous on the closed interval [0,2] and has values that are given in the table below. The equation $f(x) = \frac{1}{2}$ must have atleast two solutions in the interval [0,2] if k =

x	0	1	2
$f(x)$	1	k	2

   (a) 0 (b) $\frac{1}{2}$ (c) 1 (d) 3

4. If the length of sub-normal is equal to the length of sub-tangent at a point $(3,4)$ on the curve $y = f(x)$ and the tangent at $(3,4)$ to $y = f(x)$ meets the coordinate axis at A and B, then the maximum area of the triangle OAB, where O is origin, is
   (a) 45/2 (b) 49/2 (c) 25/2 (d) 81/2

5. If $f(x) = \frac{x^2}{2} + 3x$ where $x = 2$, $\delta x = 0.05$ then $df =$
   (a) 2.5 (b) 0.25 (c) 2.635 (d) 2.652

6. $\int \cos(\log_e x)\, dx$ is equal to:
   (a) $\frac{1}{2} x[\cos(\log_e x) + \sin(\log_e x)] + c$
   (b) $x[\cos(\log_e x) + \sin(\log_e x)] + c$
   (c) $\frac{1}{2} x[\cos(\log_e x) - \sin(\log_e x)] + c$
   (d) $x[\cos(\log_e x) - \sin(\log_e x)] + c$

7. The value of definite integral $\int_{\infty}^{0} \frac{ze^{-z}}{\sqrt{1-e^{-2z}}}\, dz$
   (a) $-\frac{\pi}{2}\ln 2$ (b) $\frac{\pi}{2}\ln 2$
   (c) $-\pi \ln 2$ (d) $\pi \ln \frac{1}{\sqrt{2}}$

8. The area bounded by the curve $y = \sin^{-1} x$ and the lines $x = 0, |y| = \frac{\pi}{2}$ is
   (a) 2 (b) 4 (c) 8 (d) 16

9. The reflection of the point (4,-13) in the line $5x + y + 6 = 0$ is
   (a) (-1, -14) (b) (3,4) (c) (1,2) (d) (-4,13)

10. If the centroid and a vertex of an equilateral triangle are (2,3) and (4,3) respectively, then the other two vertices of the triangle are

(a) $(1,3\pm\sqrt{3})$ (b) $(2,3\pm\sqrt{3})$

(c) $(1,2\pm\sqrt{3})$ (d) $(2,2\pm\sqrt{3})$

11. The locus of a point which moves so that the ratio of the length of the tangents to the circles $x^2+y^2+4x+3=0$ and $x^2+y^2-6x+5=0$ is 2 : 3 is

(a) $5x^2+5y^2-60x+7=0$

(b) $5x^2+5y^2+60x-7=0$

(c) $5x^2+5y^2-60x-7=0$

(d) $5x^2+5y^2+60x+7=0$

12. If the equation of a parabola is $x^2=-9y$, then equation of directrix and length of latus rectum are

(a) $y=-\frac{9}{4},8$ (b) $y=\frac{9}{4},9$

(c) $y=-\frac{9}{4},9$ (d) None of these

13. (2, 4) and (10, 10) are the ends of a latus rectum of an ellipse with $e=\frac{1}{2}$. The length of the major axis is

(a) $\frac{50}{3}$ (b) $\frac{40}{3}$ (c) $\frac{25}{3}$ (d) $\frac{20}{3}$

14. Real numbers $a_1,a_2,...,a_{99}$ form an arithmetic progression. Suppose that

$a_2+a_5+a_8+...+a_{98}=205.$

Find the value of $\sum_{k=1}^{99}a_k$.

(a) 516 (b) 615 (c) 415 (d) 715

15. If $|z-1|\le 3$ then the maximum value of $|z+4|$ is

(a) 3 (b) 4 (c) 8 (d) 12

16. If the coefficients of x^p and x^q (p and q are positive integers) in the expansion of $(1+x)^{p+q}$ are

(a) Equal

(b) Equal with opposite signs

(c) Reciprocal to each other

(d) Un equal

17. If the integers m and n are chosen at random from 1 to 100, then the probability that a number of the form 7^n+7^m is divisible by 5 equals

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{8}$ (d) $\frac{1}{3}$

18. If $a=2i+2j-k$ and $|xa|=1$, then $x=$

(a) $\pm\frac{1}{3}$ (b) $\pm\frac{1}{4}$ (c) $\pm\frac{1}{5}$ (d) $\pm\frac{1}{6}$

19. If $y=e^{(k+1)x}$ is a solution of differential equation $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0$, then $k=$

(a) -1 (b) 0 (c) 1 (d) 2

20. The equation of the plane which is parallel to y-axis and cuts off intercepts of length 2 and 3 from x-axis and z-axis is

(a) $3x+2z=1$ (b) $3x+2z=6$

(c) $2x+3z=6$ (d) $3x+2z=0$

21. If $\cos x-\sin x\ge 1$ and $0\le x\le 2\pi$ then the solution set for x is

(a) $\left[0,\frac{\pi}{4}\right]\cup\left[\frac{7\pi}{4},2\pi\right]$ (b) $\left[\frac{3\pi}{2},\frac{7\pi}{4}\right]\cup\{0\}$

(c) $\left[\frac{3\pi}{2},2\pi\right]\cup\{0\}$ (d) None of these

22. If $y=\log\cos\sqrt{x}$, then $\frac{dy}{dx}=$

(a) $\frac{\tan\sqrt{x}}{2\sqrt{x}}$ (b) $-\frac{\tan\sqrt{x}}{2\sqrt{x}}$

(c) $\tan\sqrt{x}$ (d) $\frac{1}{\cos\sqrt{x}}$

23. The limit of $\frac{x^2-1}{x-1}$ as x approaches 1 as a limit is :

(a) 0 (b) Indeterminate

(c) $x-1$ (d) 2

24. $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}=$

(a) $a^2b^2c^2$ (b) 4 abc
(c) $4a^2b^2c^2$ (d) $2a^2b^2c^2$

25. Let A be a 5×7 matrix, then each column of A contains
(a) 6 elements (b) 5 elements
(c) 7 elements (d) None of these

26. A tower subtends angles $\alpha, 2\alpha$, and 3α respectively at points A, B and C, all lying on a horizontal line through the foot of the tower. Then AB/BC is :
(a) $1+2\cos 2\alpha$ (b) $1+\cos 2\alpha$
(c) $1+2\cos\alpha$ (d) None of these

27. The solution set of the in equalities $3x^2-2x\le1$ and $15x^2-x\ge6$.

(a) $\left[-\infty,\frac{2}{3}\right)$ (b) $[1,\infty)$

(c) $\left[\frac{2}{3},1\right]$ (d) $\left(\frac{2}{3},1\right)$

28. The number of ways can 5 things be divided between A and B so that each receive at least one thing is
(a) 30 (b) 60 (c) 20 (d) 80

29. An urn contains 4 white and 3 red balls. Three balls are drawn with replacement from this urn. Then, the standard deviation of the number of red balls drawn is

(a) $\frac{6}{7}$ (b) $\frac{36}{49}$ (c) $\frac{5}{7}$ (d) $\frac{25}{49}$

30. Statement $(p\wedge q)\to p$ is-
(a) A tautology (b) A contradiction
(c) Neither (a) nor (b) (d) None of these

ANSWER KEY

1. c	2. b	3. a	4. b	5. b
6. a	7. a	8. a	9. a	10. a
11. d	12. b	13. b	14. b	15. c
16. a	17. a	18. a	19. c	20. b
21. c	22. b	23. d	24. c	25. b
26. a	27. c	28. a	29. a	30. a

HINTS & SOLUTIONS

1.Sol: Let x be an arbitrary element of A. Then , there exists a subset, say x, of set A such that $x\in X$.

Now, $x\subset A$

$\Rightarrow X\in P(A)$

$\Rightarrow X\in P(B)$

i.e., $X\subset B$

$\Rightarrow x\in B$

Thus, $x\in A\Rightarrow x\in B$

$\therefore\quad A\subset B \qquad (1)$

Now, let y be an arbitrary element B. Then , there exists a subset , say Y, of set B such that $y\in Y$.

Now , $Y\subset B\Rightarrow Y\in P(B)$

$\Rightarrow Y\in P(A)$

$\Rightarrow Y\subset A \qquad Y\in A$

Thus, $y\in B\Rightarrow y\in A$

$\therefore\quad B\subset A \qquad (2)$

From (1) and (2) , we get $A=B$

3.Sol: Given $f(x)$ is continuous on the closed interval $[0,2]$

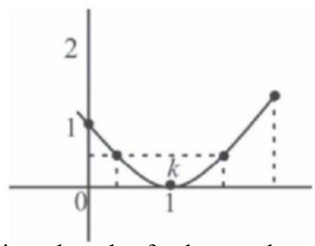
Now $\quad f(0)=1,\ f(2)=2,$ and $f(1)=k$

Using Intermediate value theorem there are two values

of x, So $f(x)=\frac{1}{2}$.

Because $\quad f(1)=0<\frac{1}{2}<f(0)$

and $\quad f(1)=0<\frac{1}{2}<f(2)$



4.Sol: Given length of subnormal and subtangent at (3,4) on the curve $y=f(x)$ are equal

i.e., $\left|\frac{y}{m}\right| = |ym|$

$|m|^2 = 1$

$m = \pm 1$

If $m=1$, equation of the tangent is

$y = x+1$

$\therefore$ area of $\Delta OAB = \frac{1}{2}1\times 1 = \frac{1}{2}$

Similarly, If $m=-1$, equation of the tangent is

$y = x+7$.

$\therefore$ Area of $\Delta OAB = \frac{1}{2}7\times 7 = \frac{49}{2}$

5.Sol: Given $f(x) = \frac{x^2}{2} + 3x, x = 2, \delta x = 0.05$

$\Rightarrow f'(x) = x+3$

We know $df = f'(x)\,\delta x$

i.e., $df = (x+3)\,0.05$

$= 5(0.05)$

$= 0.25$

6.Sol: Given that

$$I = \int \cos(\log_e^x)\,dx = \int \cos(\log_e^x)\cdot 1dx$$

$$I = \cos(\log_e^x)x - \int \frac{-\sin(\log_e^x)}{x}\,x\cdot dx$$

$$= x\cos(\log_e^x) + \int \sin(\log_e^x)\,dx$$

$$= x\cos(\log_e^x) + \sin(\log_e^x)x - \int \frac{\cos(\log_e^x)}{x}\,x\cdot dx$$

$$= x\cos(\log_e^x) + x\sin(\log_e^x) - I$$

$$\Rightarrow 2I = x\left[\cos(\log_e^x) + \sin(\log_e^x)\right]$$

$$\Rightarrow I = \frac{x}{2}[\cos(\log_e^x) + \sin(\log_e^x)]$$

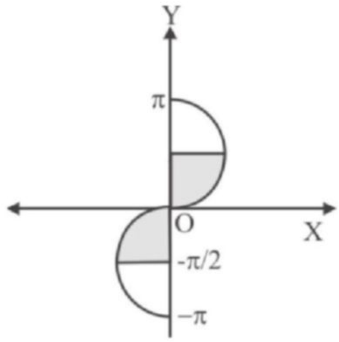
7.Sol: $I = \int_{\infty}^{0} \frac{ze^{-z}}{\sqrt{1-e^{-2z}}}\,dz$

put $e^{-z} = \sin\theta$

$$I = -\int_0^{\pi/2} \frac{\ln(\sin\theta)(-\cos\theta)\,d\theta}{\sqrt{1-\sin^2\theta}} = \int_0^{\pi/2} \ln\sin\theta\, d\theta$$

$$= \frac{-\pi}{2}\ln 2$$

8.Sol: The required area is



$$2\int_0^{\pi/2} x dy \quad \text{where } y = \sin^{-1}x \text{ i.e., } x = \sin y$$

$$= 2\int_0^{\pi/2} \sin y dy = -2[\cos y]_0^{\pi/2} = 2$$

9.Sol: Let $Q(a,b)$ be the reflection of $P(4,-13)$ in the line $5x+y+6=0$.

Then the mid-point $R\left(\frac{a+4}{2}, \frac{b-13}{2}\right)$ lies on $5x+y+6=0$.

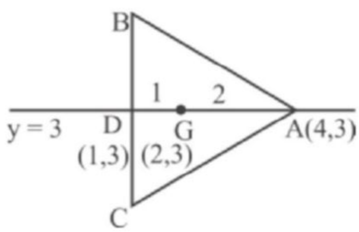
$$\therefore 5\left(\frac{a+4}{2}\right) + \frac{b-13}{2} + 6 = 0 \Rightarrow 5a+b+19=0 \quad (1)$$

Also PQ is perpendicular to $5x+y+6=0$

$$\text{Therefore } \frac{b+13}{a-4}\times\left(-\frac{5}{1}\right) = -1 \Rightarrow a-5b-69=0 \quad (2)$$

Solving (1) and (2), we get $a=-1, b=-14$.

10.Sol: G being the centroid, divides AD in the ratio 2 : 1.



Since $AG = 2$, $\therefore$ $GD = 1$

$\therefore$ Coordinates of D, using section formula, are D (1,3).

Now $AD = 1+2 = 3$,

$\therefore \quad \tan 60° = \frac{3}{BD} \quad \Rightarrow \quad BD = \sqrt{3}$

$\therefore \quad B \equiv (1, 3+\sqrt{3})$ and $C \equiv (1, 3-\sqrt{3})$.

11.Sol: Let the point be (x_1, y_1)

According to question, $\frac{\sqrt{x_1^2 + y_1^2 + 4x_1 + 3}}{\sqrt{x_1^2 + y_1^2 - 6x_1 + 5}} = \frac{2}{3}$

Squaring both sides, $\frac{x_1^2 + y_1^2 + 4x_1 + 3}{x_1^2 + y_1^2 - 6x_1 + 5} = \frac{4}{9}$

$\Rightarrow$

$9x_1 + 9y_1^2 + 36x_1 + 27 = 4x_1^2 + 4y_1^2 - 24x_1 + 20$

$\Rightarrow \quad 5x_1^2 + 5y_1^2 + 60x_1 + 7 = 0$

Hence, locus is $5x^2 + 5y^2 + 60x + 7 = 0$.

12.Sol: Given equation of parabola $x^2 = -9y$, which is of the form $x^2 = -4ay$.

i.e., focus lies on the negative direction of Y-axis.

Here, $\quad 4a = 9 \Rightarrow a = 9/4$

$\therefore$ Equation of directrix $y = a$

$\therefore \quad y = 9/4$

and length of latusrectum $= 4a = 4 \times \frac{9}{4} = 9$

13.Sol: We have length of latus rectum $\sqrt{64+36} = 10$

i.e., $\quad \frac{2b^2}{a} = 10 \Rightarrow b^2 = 5a$

but $\quad b^2 = a^2(1-e^2)$

i.e., $\quad a^2(1-e^2) = 5a$

$a^2\left(1 - \frac{1}{4}\right) = 5a$

$\Rightarrow \quad a^2\left(\frac{3}{4}\right) = 5a$

$\therefore \quad a = \frac{20}{3}$

$\therefore$ Length of major axis is $\frac{40}{3}$

14.Sol: a_{50} is the "middle" term, and hence also the mean, of both arithmetic progressions

$S = a_2 + a_5 + a_8 + ... + a_{98}$ and

$T = a_1 + a_2 + a_3 + ... + a_{99}$.

As there are 33 terms in S and 99 in T, we then know that

$S = 33_{a_{50}}, T = 99_{a_{50}} \Rightarrow T = 3S = 3 \times 205 = 615.$

15.Sol: Given $|z-1| \le 3$

we have $|z-1+b| \le |z-a| + |b|$

Now, $|z-4| = |z-a+b| \le |z-1| + |5|$

$\le \quad 3+5$

≤ 8

$\therefore$ Maximum value of $|z-4|$ is 8.

16.Sol: Coefficient of x^p in $(1+x)^{p+q}$ is ${}^{p+q}C_p$

Coefficient of x^q in $(1+x)^{p+q}$ is ${}^{p+q}C_p$

They are equal.

17.Sol: we have sample space $n(S) = 100 \times 100$

according to power cyclicity of 7, units digit $7^n = 7$ if n is of the form $4k+1, 7^n = 9$ if n is of the form $4n+2, 7^n = 3$ if n is of the form $4n+3$, and $7^n = 1$ if n is of the form $4n$

It is clearly known to us that out of 100 powers of 7 each 25 have these units digits stated above.

We know that a number with unit digit either 5 or 0 is divisible by 5. From these combinations of unit digits, there only 2 desired combinations :

(a) $7+3$

(b) $9+1$

As stated above, there are each 25 powers with unit digit 7 and 3 so number of sample points in event $= 2 \times 25^2$. We've multiplied by 2 because values of m and n can be interchanged. The same thing is true for event 2.

Required Probability is $\frac{2 \times (2 \times 25^2)}{100 \times 100} = \frac{1}{4}$.

18.Sol: $|xa| = |x||a| \Rightarrow |x|\sqrt{4+4+1} = 1 \Rightarrow x = \pm\frac{1}{3}$

19.Sol: Given $y = e^{(k+1)x}$, so that $\frac{dy}{dx} = (k+1)e^{(k+1)x}$

and $\frac{d^2y}{dx^2} = (k+1)^2 e^{(k+1)x}$

The auxiliary equation is found to be :

$$(k+1)^2 - 4(k+1) + 4 = 0$$

so $(k+1-2)^2 = 0$

so that $k = 1$.

20.Sol: Let equation of plane parallel to y-axis is,

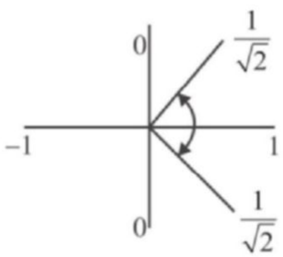
$ax + bz + 1 = 0$

Also, $2a + 1 = 0 \Rightarrow a = -\frac{1}{2}$

and $3b + 1 = 0 \Rightarrow b = -\frac{1}{3}$

$\therefore 3x + 2z = 6.$

21.Sol: $\cos\left(x + \frac{\pi}{4}\right) \geq \frac{1}{\sqrt{2}}$. The value scheme for this is shown below.



From the figure, $-\frac{\pi}{4} \leq x + \frac{\pi}{4} \leq \frac{\pi}{4}$ and

In general, $2n\pi - \frac{\pi}{4} \leq x + \frac{\pi}{4} \leq 2n\pi + \frac{\pi}{4}$

$\therefore \; 2n\pi - \frac{\pi}{2} \leq x \leq 2n\pi.$

For $n = 0, -\frac{\pi}{2} \leq x \leq 0$; for $n = 1, \frac{3\pi}{2} \leq x \leq 2\pi$.

22.Sol: Given that $y = \log\cos\sqrt{x}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos\sqrt{x}}\left(-\sin\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}} = \frac{-\tan\sqrt{x}}{2\sqrt{x}}$$

23.Sol: Solution Limits do not take the value of the limiting function at the specied value into account, so we are essentially being asked to nd the limit of $x+1$ as x approaches 1. This is simply (d) 2.

24.Sol: Given $\Delta = \begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$

$R_1 - (R_2 + R_3) \to R_1$

$$= -2\begin{vmatrix} 0 & c^2 & b^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

Now $R_2 - R_1 \Rightarrow R_2$ and $R_3 - R_1 \Rightarrow R_3$

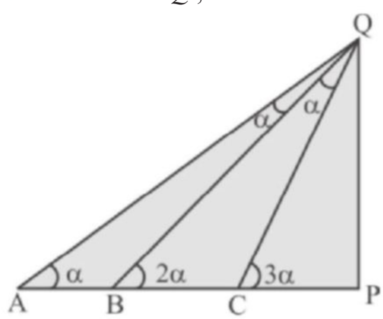
i.e., $$= -2\begin{vmatrix} 0 & c^2 & b^2 \\ b^2 & c^2 & 0 \\ c^2 & 0 & a^2 \end{vmatrix}$$

$$= -2\left\{-c^2\left(b^2a^2\right) + b^2\left(-c^2a^2\right)\right\}$$

$$= 4a^2b^2c^2$$

25.Sol: Since the matrix A is of type 5×7, therefore, it contains 5 rows and 7 columns. So, each row contains 7 elements and each column contains 5 elements.

26.Sol: From ΔABQ, we have $AB = BQ$



From ΔBCQ, we have $\frac{BC}{\sin\alpha} = \frac{BQ}{\sin 3\alpha}$

$$\Rightarrow \frac{AB}{BC} = \frac{\sin 3\alpha}{\sin\alpha} = \frac{3\sin\alpha - 4\sin^3\alpha}{\sin\alpha}$$

$$= 3 - 4\sin^2\alpha$$
$$= 1 + 2\left(1 - 2\sin^2\alpha\right)$$
$$= 1 + 2\cos 2\alpha$$

27.Sol: Given the system of inequalities

$3x^2 - 2x \le 1$ and $15x^2 - x \ge 6$

$$\Rightarrow \quad 3\left(x - \frac{1}{3}\right)(x-1) \le 0$$

and
$$15\left(x + \frac{3}{5}\right)\left(x - \frac{2}{3}\right) \ge 0$$

$$\therefore \quad x \in \left[\frac{-1}{3}, 1\right] \text{ and } x \in \left(-\infty, \frac{-3}{5}\right] \cup \left[\frac{2}{3}, \infty\right)$$

$$\Rightarrow \quad x \in \left[\frac{2}{3}, 1\right]$$

Solution set of the given system is $\left[\frac{2}{3}, 1\right]$

28.Sol: This question can be modelled as division of non identical objects.

that is $\frac{5!}{1!\,4!} + \frac{5!}{2!\,3!} + \frac{5!}{3!\,2!} + \frac{5!}{4!\,1!} = 2^5 - 2$

29.Sol: Let X denotes the number of red balls.

here, probability of getting red balls, $p = \frac{3}{7}$ and

probability of getting no red balls, $q = \frac{4}{7}$

(i) $P_1(X=0) = {}^3C_0 \left(\frac{3}{7}\right)^0 \left(\frac{4}{7}\right)^3 = \frac{64}{(7)^3}$

(ii) $P_2(X=1) = {}^3C_1 \left(\frac{3}{7}\right)^1 \left(\frac{4}{7}\right)^2 = \frac{144}{(7)^3}$

(iii) $P_3(X=2) = {}^3C_2 \left(\frac{3}{7}\right)^2 \left(\frac{4}{7}\right)^2 = \frac{108}{(7)^3}$

(iv) $P_4(X=3) = {}^3C_3 \left(\frac{3}{7}\right)^3 = \frac{27}{(7)^3}$

$$\therefore \text{ Variance } = \sum_{i=0}^{1} P_i x_i^2 - \left(\sum_{i=0}^{3} P_i x_i\right)^2$$

$$= \left[\frac{64}{(7)^3} \times 0 + \frac{144}{(7)^3} \times (1)^2 + \frac{108}{(7)^3} \times (2)^2 + \frac{27}{(7)^3} \times (3)^2\right]$$
$$- \left[\frac{64}{(7)^3} \times 0 + \frac{144}{(7)^3} \times 1 + \frac{108}{(7)^3} \times 2 + \frac{27}{(7)^3} \times 3\right]^2$$

$$= \left[0 + \frac{144}{343} + \frac{432}{343} + \frac{243}{343}\right] - \left[0 + \frac{144}{343} + \frac{216}{343} + \frac{81}{343}\right]$$

$$= \frac{819}{343} - \left(\frac{441}{343}\right)^2 = \frac{280917 - 194481}{(343)^2} = \frac{36}{49}$$

Now, standard deviation $= \sqrt{\text{variance}} = \sqrt{\frac{36}{49}} = \frac{6}{7}$

30.Sol: $\because \quad (p \wedge q) \to p$ is false

$\Rightarrow \quad (p \wedge q)$ is true and p is false which is not possible

so $(p \wedge q) \to p$ is always true i.e., it is a tautology.

CLASS XII MATHEMATICS KVPY-8

PREVIOUS YEAR QUESTIONS

Trigonometry

1. One of the solutions of the equation $8\sin^3\theta - 7\sin\theta + \sqrt{3}\cos\theta = 0$ lies in the interval **[2017]**

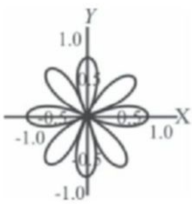
(a) $(0, 10°]$ (b) $(10°, 20°]$

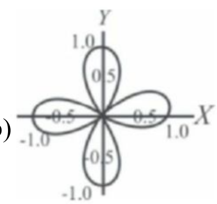
(c) $(20°, 30°]$ (d) $(30°, 40°]$

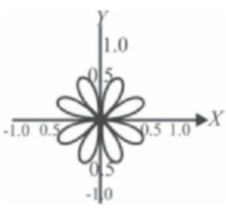
2. Consider the following parametric equation of a curve:

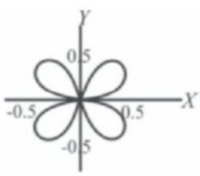
$$x(\theta) = |\cos 4\theta|\cos\theta\,;\quad y(\theta) = |\cos 4\theta|\sin\theta$$

for $0 \le \theta \le 2\pi$. Which one of the following graphs represent the curve? **[2017]**

(a) 

(b) 

(c) 

(d) 

3. The integer part of the number

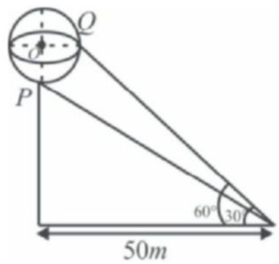
$$\sum_{k=0}^{44} \frac{1}{\cos k° \cos(k+1)°} \text{ is}$$ **[2017]**

(a) 50 (b) 52 (c) 57 (d) 59

4. Let $X = \{x \in R : \cos(\sin x) = \sin(\cos x)\}$. The number of elements in X is **[2016]**

(a) 0 (b) 2 (c) 4 (d) not finite

5. A sphere with centre O sits atop a pole as shown in the figure. An obesever on the ground is at a distance 50m from the foot of the pole. She notes that the angles of elevation from the observer to points P and Q on the sphere are $30°$ and $60°$, respectively. Then,the radius of the sphere in meters is **[2016]**



(a) $100\left(1 - \frac{1}{\sqrt{3}}\right)$ (b) $\frac{50\sqrt{6}}{3}$

(c) $50\left(1 - \frac{1}{\sqrt{3}}\right)$ (d) $\frac{100\sqrt{6}}{3}$

6. Let AB be a sector of a circle with centre O radius d, and $\angle AOB = \theta\left(< \frac{\pi}{2}\right)$, and D be a point on OA such that BD is perpendicular to OA. Let E be the midpoint of BD and F be a point on the arc AB such that EF is parallel to OA. Then the ratio of length of the arc AF to the length of the arc AB is **[2016]**

(a) $\frac{1}{2}$ (b) $\frac{\theta}{2}$

(c) $\frac{1}{2}\sin\theta$ (d) $\frac{\sin^{-1}\left(\frac{1}{2}\sin\theta\right)}{\theta}$

7. The number of real numbers λ for which the equality $\frac{\sin(\lambda\alpha)}{\sin\alpha} - \frac{\cos(\lambda\alpha)}{\cos\alpha} = \lambda - 1$, holds for all real α which are not integral multiples of $\pi/2$ is **[2015]**

(a) 1 (b) 2 (c) 3 (d) Infinite

8. Suppose $ABCDEF$ is a hexagon such that $AB = BC = CD = 1$ and $DE = EF = FA = 2$. If the vertices A, B, C, D, E, F are concylic , the radius of the circle passing through them is **[2015]**

(a) $\sqrt{\frac{5}{2}}$ (b) $\sqrt{\frac{7}{3}}$ (c) $\sqrt{\frac{11}{5}}$ (d) $\sqrt{2}$

9. Let $C(\theta) = \sum_{n=0}^{\infty} \frac{\cos(n\theta)}{n!}$. Which of the following statements is FALSE? **[2015]**

(a) $C(0).C(\pi) = 1$

(b) $C(0) + C(\pi) > 2$

(c) $C(\theta) > 0 \; for \; all \; \theta \in R$

(d) $C'(\theta) \neq 0 \; for \; all \; \theta \in R$

10. Let ABC be an acute-angled triangle and let D be the midpoint of BC. If $AB = AD$, then $\frac{\tan(B)}{\tan(C)}$ equals **[2013]**

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) 2 (d) 3

11. The angles α, β, γ of a triangle satisfy the equations $2\sin\alpha + 3\cos\beta = 3\sqrt{2}$ and $3\sin\beta + 2\cos\alpha = 1$. Then angle γ equals **[2013]**

(a) 150° (b) 120° (c) 60° (d) 30°

12. Let XY be the diameter of a semicircle with centre O. Let A be a variable point on the semicircle and B another point on the semicircle such that AB is parallel to XY. The value of $\angle BOY$ for which the inradius of triangle AOB is maximum ,is **[2013]**

13. The sum of all $x \in [0, \pi]$ which satisfy the equation $\sin x + \frac{1}{2}\cos x = \sin^2\left(x + \frac{\pi}{4}\right)$ is **[2012]**

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) π (d) 2π

14. All the points (x, y) in the plane satisfying the equation $x^2 + 2x\sin(xy) + 1 = 0$ lie on **[2011]**

(a) a pair of straight lines (b) a family of hyperbolas
(c) a parabola (d) an ellipse

15. Let $A = \{\theta \in R \mid \cos^2(\sin\theta) + \sin^2(\cos\theta) = 1\}$ and $B = \{\theta \in R \mid \cos(\sin\theta)\sin(\cos\theta) = 0\}$. Then $A \cap B$ **[2011]**

(a) is the empty set
(b) has exactly one element
(c) has more than one but finitely many elements
(d) has infinitely many elements

16. The product $(1 + \tan 1°)(1 + \tan 2°)(1 + \tan 3°) \cdots (1 + \tan 45°)$ equals **[2010]**

(a) 2^{21} (b) 2^{22} (c) 2^{23} (d) 2^{25}

ANSWER KEY

1. b	**2.** a	**3.** c	**4.** a	**5.** c
6 . d	**7.** c	**8.** b	**9.** d	**10.** d
11. d	**12.** $\frac{\sqrt{5}-1}{2}$	**13** .c	**14.** a	**15.** a
16. c				

HINTS & SOLUTIONS

1.Sol: Using $4\sin^3\theta = 3\sin\theta - \sin\theta$, rewrite the given equation

$$8\sin^3\theta - 7\cos\theta + \sqrt{3}\cos\theta = 0$$

as $6\sin\theta - 2\sin 3\theta - 7\cos\theta + \sqrt{3}\cos\theta = 0$

$\Rightarrow \quad \sqrt{3}\cos\theta - \sin\theta = 2\sin 3\theta$

Multiplying and dividing by 2, we get

$$2\sin(60 - \theta) = 2\sin 3\theta$$

i.e., $\quad \sin(60 - \theta) = \sin 3\theta$

$\Rightarrow \quad 4\theta = 60.$

i.e., $\quad \theta = 15.$

2.Sol: Conceptual

3.Sol: Expand the sequences

$$\sum_{0}^{44} \frac{1}{\cos k° + \cos(k+1)°} = \frac{1}{\cos 0° + \cos 1°} + \frac{1}{\cos 1° + \cos 2°}$$

$$+ \frac{1}{\cos 2° + \cos 3°} + \ldots + \frac{1}{\cos 44° + \cos 45°}$$

Multiplying and dividing by $\sin 1°$ on R.H.S., we get

$$= \frac{1}{\sin 1^\circ}\left[\frac{\sin 1^\circ}{\cos 0^\circ + \cos 1^\circ} + \frac{\sin 1^\circ}{\cos 1^\circ + \cos 2^\circ}\right.$$

$$\left. + \frac{\sin 1^\circ}{\cos 2^\circ + \cos 3^\circ} + \ldots + \frac{\sin 1^\circ}{\cos 44^\circ + \cos 45^\circ}\right]$$

$$= \frac{1}{\sin 1^\circ}\left[\frac{\sin(1^\circ - 0^\circ)}{\cos 0^\circ + \cos 1^\circ} + \frac{\sin(2^\circ - 1^\circ)}{\cos 1^\circ + \cos 2^\circ}\right.$$

$$\left. + \frac{\sin(3^\circ - 2^\circ)}{\cos 2^\circ + \cos 3^\circ} + \ldots + \frac{\sin(45^\circ - 44^\circ)}{\cos 44^\circ + \cos 45^\circ}\right]$$

$$= \frac{1}{\sin 1^\circ}[\tan 1^\circ - \tan 0^\circ + \tan 2^\circ - \tan 1^\circ$$

$$+ \ldots + \tan 45^\circ - \tan 44^\circ]$$

$$= \frac{1}{\sin 1^\circ}[\tan 45^\circ] = \frac{1}{0.0174524} = 57.2987$$

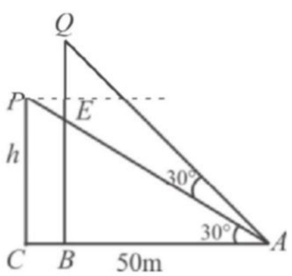
Therefore the integer part of the given sequence is 57.

4.Sol: The function $f(x) = \cos(\sin x) - \sin(\cos x)$ is even and periodic with period 2π. Therefore it is sufficies to consider $x \in [0, \pi]$. When $x \in \left[0, \frac{\pi}{2}\right]$ or $x \in \left[\frac{\pi}{2}, \pi\right]$ then $f(x) > 0$. Finally, when $0 < x < \frac{\pi}{2}$ the $\cos x$ and $\sin x$ both lie in the interval $(0,1) \subset \left(0, \frac{\pi}{2}\right)$. Therefore we also have

$$\sin(\cos x) < \cos x < \cos(\sin x) \qquad \left\{x \in \left(0, \frac{\pi}{2}\right)\right\}$$

Contradiction, and the equation has no real roots.

5.Sol: Redraw the diagram given in the question and name the points.



clearly $PE = BC = r$

now from ΔACP, we have $\tan 30^\circ = \frac{h}{50}$

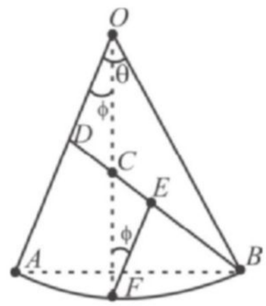
i.e., $\qquad h = \frac{50}{\sqrt{3}}$.

and from ΔABQ, we have $\tan 60^\circ = \frac{h + r}{50 - r}$.

after simpelyfing, we get $r = \frac{100}{3 + \sqrt{3}}$

i.e., $\qquad r = \frac{100(3 - \sqrt{3})}{6} = 50\left(1 - \frac{1}{\sqrt{3}}\right)$

6.Sol: $\angle AOF$ be ϕ. Then from the figure, we have



$$\begin{aligned} CF &= OF - OC \\ &= r - OD\sec\phi \\ &= r - OB\cos\theta \cdot \sec\phi \\ &= r - r\cos\theta\sec\phi \end{aligned}$$

Simillarly, $CD = r\cos\theta\tan\phi$

Now $\qquad EC + CD = ED$

i.e., $\qquad r\sin\phi = \frac{BD}{2} = \frac{r\sin\theta}{2}$

$\therefore \qquad \phi = \sin^{-1}\left(\frac{\sin\theta}{2}\right)$

$\therefore$ The required ratio is length of arc AF to the length of arc AB.

i.e., $\qquad \phi : \theta = \sin^{-1}\left(\frac{\sin\theta}{2}\right) : \theta$

7.Sol: Rewrite the given equation as

$$\frac{\sin\lambda\alpha \cdot \cos\alpha - \cos\lambda\alpha\sin\alpha}{\sin\alpha\cos\alpha} = \lambda - 1$$

$$\Rightarrow \quad \frac{\sin(\lambda-1)\alpha}{\sin\alpha\cdot\cos\alpha}=\lambda-1$$

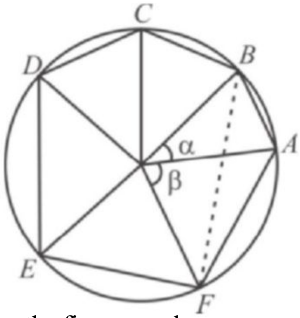
i.e., $$\frac{2\sin(\lambda-1)\alpha}{\sin 2\alpha}=\lambda-1$$

$$\Rightarrow \quad \sin(\lambda-1)\alpha=\frac{(\lambda-1)}{2}\sin 2\alpha$$

Which is possible, only for two cases, that is $\frac{\lambda-1}{2}=0$ or ±1. Which yields $\lambda=-1,0,1$.

8.Sol: Let $\angle AOB$ be α, and $\angle AOF$ be β. Then $3\alpha+3\beta=360°$.

i.e., $\alpha+\beta=120°$.



from the figure, we have

$$\angle A=\alpha+\beta=120°$$

Now $$\cos A=\cos 120°=\frac{1^2+2^2-FB^2}{2(1)(2)}$$

$$\Rightarrow \quad FB=\sqrt{7}$$

again from ΔFOB, we have

$$\cos(\alpha+\beta)=\frac{r^2+r^2-7}{2r^2}$$

$$\Rightarrow \quad \frac{-1}{2}=\frac{2r^2-7}{2r^2}$$

i.e., $$r=\sqrt{\frac{7}{3}}$$

9.Sol: Given that $C(\theta)=\sum_{n=0}^{\infty}\frac{\cos(n\theta)}{n!}$

we have $e^{i\theta}=\cos\theta+i\sin\theta$.

i.e., $\cos\theta=\text{Re}(e^{i\theta})$.

So the given expression is rewritten as

$$C(\theta)=\text{Re}\left(\sum_{n=0}^{\infty}\frac{e^{(i\theta)^n}}{n!}\right)$$

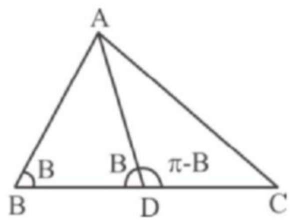
$$=\text{Re}\left(e^{i\theta}\right)=e^{\cos\theta}\cos(\sin\theta)$$

now $C(0)=e^{\cos 0°}\cos(\sin 0°)=e$ and

$$C(\pi)=e^{\cos\pi}\cos(\sin\pi)=e^{-1}=\frac{1}{e}$$

10.Sol: Given that $AB=AD$, therefore ΔABD is isosceles.

i.e., $\angle ABD=\angle ADB$



U sing M-N Rule, we have

$$(1+1)\cot(\pi-B)=1.\cot B-\cot C$$

$$\Rightarrow \quad 3\cot B=\cot C$$

i.e., $$\frac{\tan B}{\tan C}=3$$

11Sol: $2\sin\alpha+3\cos\beta=3\sqrt{2}$ (1)

$3\sin\beta+2\cos\alpha=1$ (2)

upon squaring and adding (1) and (2), we get

$$4+9+12\sin(\alpha+\beta)=19$$

i.e., $\sin(\alpha+\beta)=\frac{1}{2}\Rightarrow\alpha+\beta=150°\ or\ 30°$

If $\alpha+\beta=30°$ then $\beta=30-\alpha$

putting the value of $'\beta'$ in eq (1) and (2), we get

$7\sin\alpha+3\sqrt{3}\cos\alpha=6\sqrt{2}$ (3)

$7\cos\alpha-3\sqrt{3}\sin\alpha=2$ (4)

solving equations (3) and (4), we get

$$\Rightarrow \quad \cos\alpha=\frac{7+9\sqrt{6}}{37}=0.8<\frac{\sqrt{3}}{2}$$

$$\therefore \cos\alpha<\cos 30° \quad \{\because \alpha>30°\}$$

$$\therefore \alpha+\beta\neq 30°$$

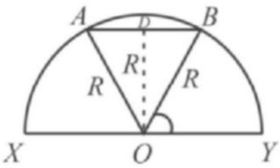
i.e., $$\alpha+\beta=150$$

Hence $\gamma = 30°$

12.Sol: Let D be the midpoint of AB.

i.e., $AD = R\cos\theta$ and $OD = R\sin\theta$

$\Rightarrow \quad AB = 2AD = 2R\cos\theta$



now, Area of $\Delta OAB = \frac{1}{2} AB \times OD$

$$= \frac{1}{2} \times 2R^2 \sin\theta \cdot \cos\theta$$

$$= R^2 \sin\theta \cdot \cos\theta$$

and, we have inradius of a ΔOAB is

$$r = \frac{\text{Area of } \Delta OAB}{\text{semiperimeter of } \Delta OAB}$$

i.e., $r = \dfrac{R^2 \sin\theta \cdot \cos\theta}{\frac{2R + AB}{2}} = \dfrac{2R^2 \sin\theta \cdot \cos\theta}{2R + 2R\cos\theta}$

$$= \frac{R\sin\theta \cdot \cos\theta}{1+\cos\theta} = \frac{R\sin 2\theta}{2(1+\cos\theta)}$$

now $\dfrac{dr}{d\theta} = \dfrac{1}{2}\left[\dfrac{2(1+\cos\theta)\cos 2\theta - \sin 2\theta(-\sin\theta)}{(1+\cos\theta)^2}\right]$

to have max in radius, $\dfrac{dr}{d\theta} = 0$

i.e., $2(1+\cos\theta)\cos 2\theta + \sin\theta \cdot \sin 2\theta = 0$

$\Rightarrow \quad 2\cos^3\theta + 4\cos^2\theta - 2 = 0$

i.e., $\quad (\cos\theta + 1)(\cos^2\theta + \cos\theta - 1) = 0$

$\Rightarrow \quad \cos^2\theta + \cos\theta - 1 = 0 \quad \{\because \cos\theta \neq -1\}$

$\therefore \quad \cos\theta = \dfrac{\sqrt{5}-1}{2}$

13.Sol: Given that $\sin x + \frac{1}{2}\cos x = \sin^2\left(x + \frac{\pi}{4}\right)$

$\Rightarrow \quad \sin x + \frac{1}{2}\cos x = \frac{1}{2}\left(1 - \cos\left(\frac{\pi}{4} + 2x\right)\right)$

i.e., $\therefore \sin x + \frac{1}{2}\cos x = \frac{1}{2}(1 + \sin 2x)$

$\Rightarrow \quad 2\sin x + \cos x = 1 + \sin x + \cos x$

$\Rightarrow \quad (1 - \cos x)(1 - 2\sin x) = 0$

$\therefore 1 - \cos x = 0$ or $1 - 2\sin x = 0$

i.e., $\cos x = 1$ or $\sin x = \frac{1}{2}$

$\therefore x = 0$, and $x = \frac{\pi}{6}, \frac{5\pi}{6}$

Hence sum of all values of x is

$$0 + \frac{\pi}{6} + \frac{5\pi}{6} = \pi$$

14.Sol: Rewrite the given equation as

$$\frac{x^2+1}{2x} + \sin(xy) = 0$$

we know, $-1 \le \sin\theta \le 1$.

i.e., $\quad -1 \le -\left(\dfrac{x^2+1}{2x}\right) \le 1$

solving inequalities, we get $(x+1)^2 \ge 0$ and $(x-1)^2 \le 0$ which yields two possible values for x. i.e., ... $-1, 1$.

Hence the locus is pair of staright lines.

15.Sol: for $A \cap B$

$\cos(\sin\theta) = 1 \; or \; -1 \;\&\; \sin(\cos\theta) = 0$

which is not possible

or $\cos(\sin\theta) = 0 \;\&\; \sin(\cos\theta) = 1$ or -1

also not possible

so $A \cap B$ is an empty set

16.Sol: Given that

$$(1+\tan 1°)(1+\tan 2°)...(1+\tan 45°)$$

we have, if $A + B = 45$, then

$$(1+\tan A)(1+\tan B) = 2$$

$\therefore$ The given expression is rewritten as

$$\{(1+\tan 1°)(1+\tan 44°)\}\{(1+\tan 2°)(1+\tan 43°)\}$$
$$...\{\tan 45°\}$$

$= 2 \times 2 \times ...23$ times $= 2^{23}$

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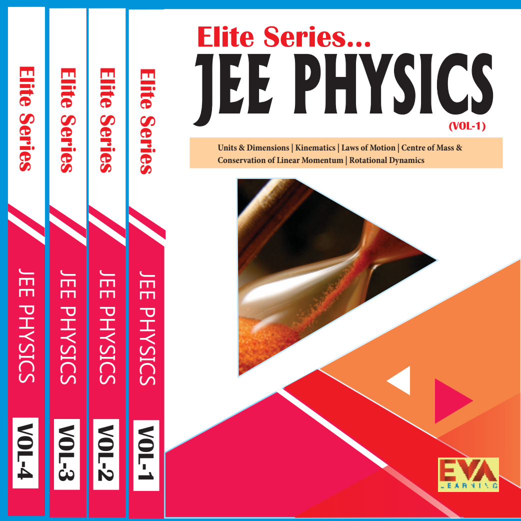

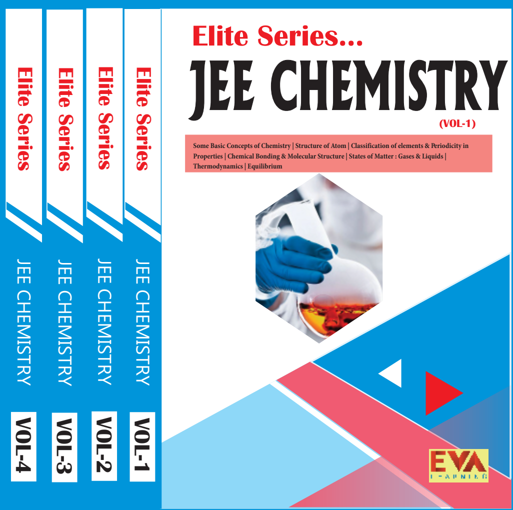

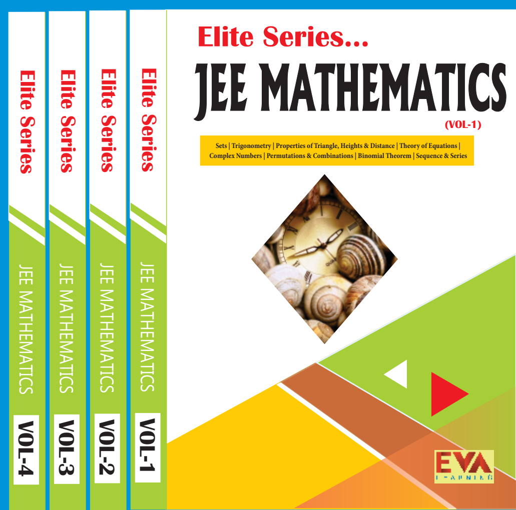

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